Thursday Dec 17:

Percolation: I a lattice, for today equilateral

Usually draw "fat vertices" hexagon picture

Color vertices black w/ prob p, white w/ prob 1-p.

Interested in connected clusters.

Let \( \Theta(p) \) be prob that a fixed vertex is

part of an infinite black cluster.

Analogy of increasing porous rock in water.

\[ \Theta(p) = \begin{cases} 0 & p < p_c \\ 1 & p > p_c \end{cases} \]

(\( p_c \) as talked about last time)

Thm: For a fixed lattice \( Z \), \( \exists p_c \in (0,1) \) s.t.

\( \Theta(p) \) increasing function

in \( p \) \( \theta(0) = 0 \)

\( \theta(1) = 1 \)

Pf: \( \theta(p) = \exists p > 0 \) w/ \( \Theta(p) = 0 \).

Fix some vertex at origin. Let

\[ \sigma(n) = \text{number of self-avoiding paths in } Z \]

of length \( n \) starting at 0

Note \( \sigma(n) \leq 6 \cdot 5^{n-1} \) and \( \sigma(n) \geq 6 \cdot 2^n \)

\[ \lambda = \lim_{n \to \infty} \sigma(n)^{1/n} \quad \lambda > 0 \]

is the exponential growth rate of \( \sigma(n) \).

Now:

\[ p \leq p_c \quad \text{is part of an infinite cluster} \quad \forall \exists \text{ some all black path of length starting at the origin} \]

\[ p \sigma(n) \leq p \cdot 6 \cdot 5^{n-1} \]
If \( p < \frac{1}{5} \), then \( p^n 6^n \to 0 \) as \( n \to 0 \).

So for \( p < \frac{1}{5} \), \( \Theta(p) = 0 \) as required.

**B:** \( \exists p < 1 \) w/ \( \Theta(p) > 0 \). Suppose 0 is not part of an infinite black cluster. Then a white ring around 0: circle of white.

Let \( p(n) = \# \) of paths surrounding 0 of length \( n \),
\[
\leq n \cdot 6^{(n-1)} \leq n 6.5^{n-1}
\]

White ring must pass through here. \( P \) if 0 not in inf cluster:

\[
\sum_{n=0}^{\infty} (1-p)^n p(n) = \sum_{n=0}^{\infty} (1-p)^n n \cdot 6.5^{n-1}
\]

for 1-\( p \) small enough \( \frac{p}{12} \leq \frac{1}{2} \). Thus \( \Theta \) (such \( p \)) > 0.

Now that we have A and B, this follows by taking:

\[
P_c = \sup \{ p | \Theta(p) = 0 \}, \text{ by monotonicity.}
\]

*From Kesten 1980* \( P_c = \frac{1}{2} \).

**Then:** Let \( U \subseteq \mathbb{C} \) be a region bounded by a closed curve with marked intervals A, B. Set

\[
\Pi(p) = \lim_{\text{mesh size} \to 0} P \{ \exists \text{a black path from A joining A to B} \}
\]

Then
\[
\Pi(p) = \begin{cases} 
0 & \text{if } p < P_c \\
1 & \text{if } p > P_c 
\end{cases}
\]
Of 2nd term \( (p > p_c \Rightarrow \text{crossing prob 0}) \) is easy direction.

Assuming 2nd term, can calculate \( p_c \) using symmetry.

Note if a black path joining \( A \to B \) \(?=\) a white path joining \( C \to D \).

Thus \( P\{\text{black A to B}\} + P\{\text{white C to D}\} = 1 \)

At \( p = 1/2 \), these are same, so \( P\{\text{black A to B}\} = 1/2 \).

\( \Rightarrow p_c = 1/2 \). (Comment on John Ward, hex, etc.)

Critical Percolation: Crossing prob at \( p_c \).

M. Aizenman: Conjecture that crossing prob is conformally invariant.

\[ f(B) \text{ conformal } B' \]

In particular, in figure

\[ P\{\text{crossing A to B}\} = \frac{P\{\text{crossing A' to B'}\}}{P\{\text{crossing A' to B'}\}} \]

J. Cardy 1988: Conjectured invariance lets you compute

the crossing probs explicitly.

Crossing prob of \((1,1)\) rectangle is

\[
\frac{\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{4}{3}\right) \Gamma\left(\frac{1}{3}\right)} \sum_{\nu=0}^{\nu_{\nu}} F_4 \left( \frac{1}{3}, \frac{2}{3}, \frac{4}{3}; \nu \right)
\]

where

\[
\nu = \sin^2 \Theta = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}
\]

\[
\Gamma'(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad \Gamma'(n) = (n-1)!
\]

\( F_4 \) = hypergeometric function.
Carlson's version:

\[
\text{unit side equilateral triangle}
\]

\[F\text{ crossing } A \rightarrow B \bar{f} = x.\]

1994: Langlands, Pauliot, Saint-Aubin; verified Selberg's formula approx in computer experiments.

2001: S. Smirnov proves conformal invariance for \( \Omega = \text{equilateral lattice}. \)

Idea: Consider a mesh of size \( S. \)

Define \( H^S(\cdot, p) = \)

\[\text{Prob}(\exists \text{ a black path } \Delta_{sp}).\]

For \( p = \mathbb{R}^2 \) this is the crossing formula for Carlson's model.

Shows that as \( S \to 0, H^S \to H. \) Smirnov shows:

1. \( H \) is harmonic \((\nabla^2 H = 0)\)
2. \( H(\Delta) = 1 \quad H(\Delta) = 0 \)
3. \( \text{on } \Delta \text{ the sum of } \nabla H \text{ which is } H \text{ to } \Delta \)

\[\Rightarrow H \text{ is linear for set } c.\]

Do in general using invariance of harmonic for under conformal maps.

The End.