Thursday, Dec 12

Conformal Mapping: \( U \text{ open} \subset \mathbb{C} \). Then \( f: U \to \mathbb{C} \)

is conformal if the derivative at every \( pt \) preserves angles.

\[ f \text{'s } \text{non-zero.} \]

Ex: Euclidean motions: dilatation: \( z \to \frac{1}{z} \) (inversion in \( 0 \))

degenerately, holomorphic maps.

Non Ex: \( (x, y) \to (2x, y) \).

Another characterization: Conformal = infinitesimal

circles go to infinitesimal circles.

Riemann Mapping Theorem: Let \( U \) be a proper open subset of \( \mathbb{C} \),

which is simply connected.

Then \( f \) a conformal map

\[ f: U \to D = \{ z \in \mathbb{C} \mid |z| < 1 \} \]

which is \( 1:1 \) and onto.

Note: The pt is not topological.

Note slides, discuss

Brownian Motion in 2D: \( U_t, V_t \) indep Brownian motions

\[ W_t = U_t + iV_t \]

Last time: Rotational invariance.
Am I being saw if \( U \) is Brownian motion, so is
\[
W_t = c \ U_t c^{-2t}, \text{ i.e., is our under dilatation after rescaling time.}
\]
Now in 2-d consider
\[
W'_t = c \ W_t
\]
By above, this is just 2-d Brownian motion, except that we've changed how fast things move.

Let \( U \subseteq \mathbb{C} \) be open, \( p \in U \). Consider constrained Brownian motion in \( U \) starting at \( p \).

Stops when hits the boundary.

Let \( f : U \rightarrow V \) be conformal.
Then \( f(W_t) \) is a time rescaled Brownian motion.

Why care? Additions of symmetries in the limit lead to greater understanding. (Refer back to slide)

Ex: Hitting probabilities.

Aside: Conformal invariance of Brown motion gives another proof of the fundamental of

Algebra.
Percollation: Start w/ a lattice $\mathbb{L}$, e.g., $\mathbb{Z}^2$.
Color vertices black and white w/ prob $p$ and $1-p$. Interested in regions of light/dark.

- Model for liquids in porous media.
- Undergo phase transition at a critical probability.

Simple question: Crossing probability.

Does there exist a black path from left to right?

Let $\Pi_k^n(p)$ be this prob for our $n \times n$ square lattice w/ black prob $p$.

**Claim:** For fixed $n$, $\Pi_k^n(p)$ is an increasing func of $p$.

**Claim:** Else, conduct of a random media.

New graph of $\Pi_k^n(p)$

If we take the limit as $n \to \infty$, there is a critical probability $p_c$ at which the crossing prob jumps from 0 to 1.

For square lattice

$$p_c \approx 0.59 \text{ (no exact formula is known!)}$$
Kesten (1980) For any lattice, there is a critical prob $p_c$ with transition prob seen above. $\pi_n$.

Q: At $p_c$ what is $\pi_n$? How about for a rectangle? A general region?

Next time: Conformal invariance of percolation (Smirnov 2001)

Cardy's crossing formula.

$p_c = \frac{1}{2}$ for triangular lattice (and John Nash).
Figure 2.1a. Configurations on the square cube $S_{16}$ for percolation by sites.
to larger values of $n$. ...32, 64, and 128. Larger slopes around $p$ correspond.

Figure 2.1c. The curves $u_n(d)$ for $n = 2, 4, 8, 16$. Crossing Probability. $d$