1. Consider the ellipsoid with implicit equation

\[ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \]

(a) Parametrize this ellipsoid.

**Solution.** One could use the parametrization

\[ x = a \sin \phi \cos \theta, \quad y = b \sin \phi \sin \theta, \quad z = c \cos \phi, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi. \]

(b) Set up, but do not evaluate, a double integral that computes its surface area.

**Solution.** Since \( \mathbf{r}(\phi, \theta) = (a \sin \phi \cos \theta, b \sin \phi \sin \theta, c \cos \phi) \), one has

\[ \mathbf{r}_\phi = (a \cos \phi \cos \theta, b \cos \phi \sin \theta, -c \sin \phi), \quad \mathbf{r}_\theta = (-a \sin \phi \sin \theta, b \sin \phi \cos \theta, 0), \]

so

\[ \mathbf{r}_\phi \times \mathbf{r}_\theta = (bc \sin^2 \phi \cos \theta, ac \sin^2 \phi \sin \theta, ab \sin \phi \cos \theta). \]

Therefore

\[ |\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sqrt{b^2 c^2 \sin^4 \phi \cos^2 \theta + a^2 c^2 \sin^4 \phi \sin^2 \theta + a^2 b^2 \sin^2 \phi \cos^2 \phi}, \]

and the surface area is computed by

\[
\text{Area} = \int_{0}^{2\pi} \int_{0}^{\pi} |\mathbf{r}_\phi \times \mathbf{r}_\theta| d\phi d\theta
= \int_{0}^{2\pi} \int_{0}^{\pi} \sqrt{b^2 c^2 \sin^4 \phi \cos^2 \theta + a^2 c^2 \sin^4 \phi \sin^2 \theta + a^2 b^2 \sin^2 \phi \cos^2 \phi} d\phi d\theta.
\]

2. Let

\[ \mathbf{r}(u, v) = ((2 + \cos u) \cos v, (2 + \cos u) \sin v, \sin u), \]

where \( 0 \leq u \leq 2\pi \) and \( 0 \leq v \leq 2\pi \).

(a) Sketch the surface parametrized by this function.

**Solution.** The sketch of the surface is as follows.
(b) Compute its surface area.

**Solution.** By the parametrization, one has

\[ r_u = (-\sin u \cos v, -\sin u \sin v, \cos u), \]
\[ r_v = (- (2 + \cos u) \sin v, (2 + \cos u) \cos v, 0), \]

and so

\[ r_u \times r_v = (- (2 + \cos u) \cos u \cos v, - (2 + \cos u) \cos u \sin v, -(2 + \cos u) \sin u). \]

Therefore \(|r_u \times r_v| = 2 + \cos u\), and the surface area is computed by

\[
\text{Area} = \int_0^{2\pi} \int_0^{2\pi} |r_u \times r_v| \, du \, dv = \int_0^{2\pi} \int_0^{2\pi} (2 + \cos u) \, du \, dv = 8\pi^2.
\]

3. Consider the surface integral

\[
\iint_{\Sigma} z \, dS
\]

where \(\Sigma\) is the surface with sides \(S_1\) given by the cylinder \(x^2 + y^2 = 1\), \(S_2\) given by the unit disk in the \(xy\)-plane, and \(S_3\) given by the plane \(z = x + 1\). Evaluate this integral as follows:

(a) Parametrize \(S_1\) using \((\theta, z)\) coordinates.

**Solution.** One can parametrize \(S_1\) by

\[ x = \cos \theta, \ y = \sin \theta, \ z = z, \quad 0 \leq \theta \leq 2\pi, \ 0 \leq z \leq \cos \theta + 1. \]

(b) Evaluate the integral over the surface \(S_2\) without parametrizing.

**Solution.** Since \(z = 0\) on \(S_2\), we know \(\iint_{S_2} z \, dS = 0\).
(c) Parametrize $S_3$ in Cartesian coordinates and evaluate the resulting integral using polar coordinates.

**Solution.** One can parametrize $S_3$ in Cartesian coordinates

$$x = x, \quad y = y, \quad z = x + 1, \quad -1 \leq x \leq 1, \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}.$$ 

Now we move to evaluate the integral $\iiint_{S} z \, dS$. Obviously

$$\iiint_{S} z \, dS = \iiint_{S_1} z \, dS + \iiint_{S_2} z \, dS + \iiint_{S_3} z \, dS = I_1 + I_2 + I_3.$$

To estimate $I_1$, using the parametrization in (a), one has

$$\mathbf{r}(\theta, z) = \langle \cos \theta, \sin \theta, z \rangle.$$

Then

$$\mathbf{r}_\theta = \langle -\sin \theta, \cos \theta, 0 \rangle, \quad \mathbf{r}_z = \langle 0, 0, 1 \rangle,$$

and

$$\mathbf{r}_\theta \times \mathbf{r}_z = \langle \cos \theta, \sin \theta, 0 \rangle.$$

So $|\mathbf{r}_\theta \times \mathbf{r}_z| = 1$, and

$$I_1 = \int_0^{2\pi} \int_0^{\cos \theta + 1} z \, dz \, d\theta = \int_0^{2\pi} \frac{(\cos \theta + 1)^2}{2} \, d\theta$$

$$= \int_0^{2\pi} \frac{\cos^2 \theta + 2 \cos \theta + 1}{2} \, d\theta = \frac{3\pi}{2}.$$

In (b) we know $I_2 = 0$. To evaluate $I_3$, by the parametrization in (c), one has

$$\mathbf{r}(x, y) = \langle x, y, x + 1 \rangle, \quad -1 \leq x \leq 1, \quad -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2},$$

and so

$$\mathbf{r}_x = \langle 1, 0, 1 \rangle, \quad \mathbf{r}_y = \langle 0, 1, 0 \rangle, \quad \mathbf{r}_x \times \mathbf{r}_y = \langle -1, 0, 1 \rangle.$$

Thus $|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{2}$, and the surface integral is

$$I_3 = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x + 1)\sqrt{2} \, dy \, dx = \iint_{x^2+y^2 \leq 1} (x+1)\sqrt{2} \, dy \, dx.$$

To evaluate this integral, one can use the polar coordinates

$$x = r \cos \theta, \quad y = r \sin \theta, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$$ 

Therefore,

$$I_3 = \int_0^{2\pi} \int_0^1 (r \cos \theta + 1)\sqrt{2} \, r \, dr \, d\theta = \sqrt{2}\pi.$$

Adding up all three integrals, one gets 

$$\iiint_{S} z \, dS = I_1 + I_2 + I_3 = \frac{3\pi}{2} + \sqrt{2}\pi.$$
4. Let \( C \) be the circle in the plane with equation \( x^2 + y^2 - 2x = 0 \).

(a) Parametrize \( C \) as follows. For each choice of a slope \( t \), consider the line \( L_t \) whose equation is \( y = tx \). Then the intersection \( L_t \cap C \) of \( L_t \) and \( C \) contains two points, one of which is \((0, 0)\). Find the other point of intersection, and call its \( x \)– and \( y \)–coordinates \( x(t) \) and \( y(t) \). Compute a formula for \( \mathbf{r}(t) = (x(t), y(t)) \). Check your answer with your TA.

**Solution.** Bring \( y = tx \) into \( x^2 + y^2 - 2x = 0 \), then one has
\[
x^2 + t^2 x^2 - 2x = 0,
\]
and it is easy to get \( x = \frac{2}{1+t^2} \), and then \( y = \frac{2t}{1+t^2} \). Thus \( \mathbf{r}(t) = \left( \frac{2}{1+t^2}, \frac{2t}{1+t^2} \right) \).

(b) Suppose that \( t = \frac{p}{q} \) is a rational number. Show that \( x(p/q) \) and \( y(p/q) \) are also rational numbers. Explain how, by clearing denominators in \( x(p/q) - 1 \) and \( y(p/q) \), you can find a a triple of integers \( U, V, \) and \( W \) for which \( U^2 + V^2 = W^2 \).

**Solution.** Plug \( t = \frac{p}{q} \) into the the parametrization, one gets
\[
x(p/q) = \frac{2q^2}{p^2 + q^2}, \quad y(p/q) = \frac{2pq}{p^2 + q^2},
\]
and both of them are rational numbers. Since \((x-1)^2 + y^2 = 1\), and \( x(p/q) - 1 = \frac{q^2 - p^2}{p^2 + q^2} \), then one has
\[
\left( \frac{q^2 - p^2}{p^2 + q^2} \right)^2 + \left( \frac{2pq}{p^2 + q^2} \right)^2 = 1.
\]
By setting
\[
U = q^2 - p^2, \quad V = 2pq, \quad W = p^2 + q^2,
\]
one has \( U^2 + V^2 = W^2 \).

(c) Compute \( \int_C \frac{1}{2} \langle -y, x \rangle \cdot d\mathbf{r} \) using your parametrization above.

**Solution.** Since \( \mathbf{r} = \left( \frac{-2}{1+t^2}, \frac{2t}{1+t^2} \right) \), one has \( \mathbf{r}' = \left( -\frac{4t}{(1+t^2)^2}, \frac{2-2t^2}{(1+t^2)^2} \right) \). Then
\[
\int_C \frac{1}{2} \langle -y, x \rangle \cdot d\mathbf{r} = \int_{-\infty}^{\infty} \frac{1}{2} \left\langle -\frac{2t}{1+t^2}, \frac{2}{1+t^2} \right\rangle \cdot \left\langle -\frac{4t}{(1+t^2)^2}, \frac{2-2t^2}{(1+t^2)^2} \right\rangle \, dt
\]
\[
= \int_{-\infty}^{\infty} \frac{2}{(1+t^2)^2} \, dt = \pi.
\]