1. Consider the region \( R \) in \( \mathbb{R}^2 \) shown below at right. In this problem, you will do a change of coordinates to evaluate:

\[
\iint_{R} x - 2y \, dA
\]

(a) Find a linear transformation \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) which takes the unit square \( S \) to \( R \).

Write you answer both as a matrix \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) and as \( T(u, v) = (au + bv, cu + dv) \), and check your answer with the instructor.

**SOLUTION:**

\( T(u, v) = (2u + v, u + 3v) \). In matrix form,

\[
\begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}
\]

(b) Compute \( \iint_{R} x - 2y \, dA \) by relating it to an integral over \( S \) and evaluating that. Check your answer with the instructor.

**SOLUTION:**

The Jacobian of \( T \) is

\[
det \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} = 6 - 1 = 5
\]

So

\[
\iint_{R} x - 2y \, dA = \iint_{S} [(2u + v) - 2(u + 3v)]5 \, dA
\]

\[
= \int_{0}^{1} \int_{0}^{1} -25v \, du \, dv = \left[ -\frac{25v^2}{2} \right]_{0}^{1} = -\frac{25}{2}
\]
2. Another simple type of transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a translation, which has the general form $T(u, v) = (u + a, v + b)$ for a fixed $a$ and $b$.

(a) If $T$ is a translation, what is its Jacobian matrix? How does it distort area?

SOLUTION:
If $T(u, v) = (u + a, v + b)$ where $a$ and $b$ are constants, then the Jacobian is

$$\det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$  

So $T$ does not distort areas.

(b) Consider the region $S = \{u^2 + v^2 \leq 1\}$ in $\mathbb{R}^2$ with coordinates $(u, v)$, and the region $R = \{(x - 2)^2 + (y - 1)^2 \leq 1\}$ in $\mathbb{R}^2$ with coordinates $(x, y)$.

Make separate sketches of $S$ and $R$.

SOLUTION:

(c) Find a translation $T$ where $T(S) = R$.

SOLUTION:
$T(u, v) = (u + 2, v + 1)$

(d) Use $T$ to reduce $\iint_R x \, dA$ to an integral over $S$, and then evaluate that new integral using polar coordinates.

SOLUTION:
The Jacobian of $T$ is just 1, as noted in part (a). So we have

$$\iint_R x \, dA = \iint_S (u + 2) \, dA$$

Converting the second integral above to polar we have

$$\iint_S (u + 2) \, dA = \int_0^{2\pi} \int_0^1 (r \cos \theta + 2) r \, dr \, d\theta = \int_0^{2\pi} \left[ \frac{r^3 \cos \theta}{3} \right]_0^1 d\theta + 2\pi \left[ r^2 \right]_0^1$$

$$= \frac{1}{3} \int_0^{2\pi} \cos \theta \, d\theta + 2\pi = \frac{1}{3} \cdot 2\pi + 2\pi = 2\pi$$
3. Consider the region $R$ shown below. Here the curved left side is given by $x = y - y^2$. In this problem, you will find a transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ which takes the unit square $S = [0, 1] \times [0, 1]$ to $R$.

![Diagram of region R]

(a) As a warm up, find a transformation that takes $S$ to the rectangle $[0,2] \times [0,1]$ which contains $R$.

**SOLUTION:**

$L(u, v) = (2u, v)$

(b) Returning to the problem of finding $T$ taking $S$ to $R$, come up with formulas for $T(u, 0), T(u, 1), T(0, v),$ and $T(1, v)$. Hint: For three of these, use your answer in part (a).

**SOLUTION:**

$T(u, 0) = (2u, 0)$

$T(u, 1) = (2u, 1)$

$T(1, v) = (2, v)$

$T(0, v) = (v - v^2, v)$

(c) Now extend your answer in (b) to the needed transformation $T$. Hint: Try “filling in” between $T(0, v)$ and $T(1, v)$ with a straight line.

**SOLUTION:**

$T(u, v) = (2u + v(1 - v)(1 - u), v)$

(d) Compute the area of $R$ in two ways, once using $T$ to change coordinates and once directly.

**SOLUTION:**

To change coordinates we compute the Jacobian

$$J(T) = \det \begin{pmatrix} 2 - v(1 - v) & (1 - 2v)(1 - u) \\ 0 & 1 \end{pmatrix} = 2 - v(1 - v)$$

So we have the area of $R$ given by

$$\iint_R \, dx \, dy = \int_0^1 \int_0^1 2 - v(1 - v) \, du \, dv = 11/6$$

Computing directly we have the area of $R$ given by

$$\int_0^1 2 - (y^2 - y) \, dy = 11/6$$
4. If you get this far, evaluate the integrals in Problems 1 and 2 directly, without doing a change of coordinates. It’s a fun-filled task…

**SOLUTION:**

For the integral in problem one, use the order $dy \, dx$. We need to split the double integral into three parts. The result is

\[
\int_{R} x - 2y \, dA = \int^{1}_{0} \int_{x/2}^{3x} x - 2y \, dy \, dx + \int^{2}_{1} \int_{x/2}^{x/2+5/2} x - 2y \, dy \, dx + \int^{3}_{2} \int_{x/2}^{x/2+5/2} x - 2y \, dy \, dx
\]

Evaluating this is not difficult but it is tedious. We leave it to the interested student. You should get $-25/2$.

For the integral in problem two, again use the order $dy \, dx$. We just need one double integral.

\[
\int_{R} x \, dA = \int^{3}_{1} \int_{1+\sqrt{1-(x-2)^2}}^{1-\sqrt{1-(x-2)^2}} x \, dy \, dx
\]

\[
= \int^{3}_{1} 2x \sqrt{1 - (x-2)^2} \, dx
\]

This integral can be evaluated by making the substitution $x - 2 = \sin u$, yielding the integral

\[
\int_{-\pi/2}^{\pi/2} (2\sin u + 4) \cos^2 u \, du
\]

Now split this in two pieces as

\[
\int_{-\pi/2}^{\pi/2} 2\sin u \cos^2 u \, du + \int_{-\pi/2}^{\pi/2} 4 \cos^2 u \, du
\]

The first is the integral of an odd function over an interval which is symmetric about the $y$ axis so it is 0. The second can be evaluated by using the trig identity $\cos^2 u = (1 + \cos 2u)/2$. This gives

\[
\int_{-\pi/2}^{\pi/2} 4 \cos^2 u \, du = \int_{-\pi/2}^{\pi/2} 4(1 + \cos 2u)/2 \, du = 2\pi.
\]