Thursday, October 24 * Solutions * Introduction to multiple integrals

1. Evaluate the following integral by reversing the order of integration:

\[ \int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1} \, dx \, dy. \]

(Hint: When you change to \( dx \, dy \), be sure to also change the bounds of integration.)

**SOLUTION:**

We are integrating over the region below:

\[ y = x^2, \quad y = 0, \quad x = 1. \]

Changing the order of integration we get

\[ \int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1} \, dx \, dy = \int_0^1 \int_0^{x^2} \sqrt{x^3 + 1} \, dy \, dx \]

\[ \int_0^1 \int_0^{x^2} \sqrt{x^3 + 1} \, dy \, dx = \int_0^1 x^2 \sqrt{x^3 + 1} \, dx = \frac{2}{9}[(x^3 + 1)^{3/2}]_0^1 = \frac{2}{9}(2^{3/2} - 1). \]
2. Consider the region bounded by the curves determined by \(-2x + y^2 = 6\) and \(-x + y = -1\).

(a) Sketch the region \(R\) in the plane.

**SOLUTION:**

\[
\begin{align*}
    & 0 & \quad \text{and} & \quad & 5, 4 & \quad \text{and} & \quad & -1, -2 & \quad \text{and} & \quad & -2, 0, 2, 4, 6 & \quad \text{and} & \quad & 0, 1, 2, 3, 4, 5
\end{align*}
\]

(b) Set up and evaluate an integral of the form \(\iint_R dA\) that calculates the area of \(R\).

**SOLUTION:**

\[
\begin{align*}
    \int_{-2}^{4} \int_{\frac{y^2-6}{2}}^{y+1} dx \, dy &= \int_{-2}^{4} y + 1 - \frac{y^2 - 6}{2} \, dy = \left[-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right]_{-2}^4 = 18
\end{align*}
\]
3. Consider the region $R$ in the first quadrant which lies above the $x$-axis and between the circles of radius 1 and 2 centered at $(0,0)$. Without using polar coordinates, evaluate 

$$\iint_R y\,dA.$$ 

**SOLUTION:** Notice that both the function $y$ and the region $R$ are symmetric about the $y$-axis, so we can integrate $y$ over the half of $R$ which lies in the first quadrant (Call this $R'$) and double our answer. $R'$ is shown below.

Break up $R'$ into two parts $A$ and $B$ as above. Integrating, we obtain

$$\iint_R y\,dA = \iint_A y\,dA + \iint_B y\,dA = \int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} y\,dy\,dx + \int_1^2 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} y\,dy\,dx$$

$$= \int_0^1 \frac{y^2}{2} \sqrt{4-x^2}\,dx + \int_1^2 \frac{y^2}{2} \sqrt{4-x^2}\,dx = \int_0^1 \frac{1}{3} \,dx + \int_1^2 \frac{1}{2}(4-x^2) \,dx$$

$$= \frac{7}{3}$$

Now double this value to get $14/3$, which is the integral over the entire region $R$. 

![Diagram of region R with regions A and B]
4. Evaluate
\[
\int_{-2}^{0} \int_{0}^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx.
\]

Hint: don't do it directly.

**SOLUTION:**

The region over which we are integrating is:

Converting to polar we get
\[
\int_{-2}^{0} \int_{0}^{\sqrt{4-x^2}} (x^2 + y^2) \, dy \, dx = \int_{\pi/2}^{\pi} \int_{0}^{2} (r^2) r \, dr \, d\theta = 2\pi
\]
5. The function \( P(x) = e^{-x^2} \) is fundamental in probability.

(a) Sketch the graph of \( P(x) \). Explain why it is called a “bell curve.”

**SOLUTION:**

(b) Compute \( I = \int_{-\infty}^{\infty} e^{-x^2} \, dx \) using the following brilliant strategy of Gauss.

i. Instead of computing \( I \), compute \( I^2 = \left( \int_{-\infty}^{\infty} e^{-x^2} \, dx \right) \left( \int_{-\infty}^{\infty} e^{-y^2} \, dy \right) \).

ii. Rewrite \( I^2 \) as an integral of the form \( \iint_R f(x, y) \, dA \) where \( R \) is the entire Cartesian plane.

**SOLUTION:**

\[
I^2 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} \, dy \, dx
\]

iii. Convert that integral to polar coordinates.

**SOLUTION:**

\[
I^2 = \int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} \, r \, dr \, d\theta
\]

iv. Evaluate to find \( I^2 \). Deduce the value of \( I \).

**SOLUTION:**

\[
\int_{0}^{2\pi} \int_{0}^{\infty} e^{-r^2} \, r \, dr \, d\theta = 2\pi \int_{0}^{\infty} r e^{-r^2} \, dr = 2\pi \lim_{t \to \infty} \int_{0}^{t} r e^{-r^2} \, dr = 2\pi \lim_{t \to \infty} \left[ -\frac{1}{2} e^{-r^2} \right]_{0}^{t} = \pi \lim_{t \to \infty} (-e^{-t^2} + 1) = \pi
\]

So \( I = \sqrt{\pi} \).
6. Compute $\int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2} \, dx \, dy$.

**SOLUTION:**

As in the previous problem, let's convert to polar coordinates.

$$
\int_0^\infty \int_0^\infty \frac{1}{(1 + x^2 + y^2)^2} \, dx \, dy = \int_0^{\pi/2} \int_0^\infty \frac{r}{(1 + r^2)^2} \, dr \, d\theta = \frac{\pi}{2} \int_0^\infty \frac{r}{(1 + r^2)^2} \, dr
$$

This is an improper integral, so

$$
\frac{\pi}{2} \int_0^\infty \frac{r}{(1 + r^2)^2} \, dr = \frac{\pi}{2} \lim_{t \to \infty} \int_0^t \frac{r}{(1 + r^2)^2} \, dr = \frac{\pi}{4} \lim_{t \to \infty} \left[ -\frac{1}{1 + t^2} \right]_0^t
$$

$$
= \frac{\pi}{4} \lim_{t \to \infty} \left( -\frac{1}{1 + t^2} + 1 \right) = \frac{\pi}{4}
$$