More on Stokes' Theorem

1. Let \( \mathbf{F} = \langle y^2, x^2, z^2 \rangle \). Show that

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}
\]

for any two closed curves as shown lying on a cylinder about the z-axis.

\[
\text{SOLUTION:}
\]

Let \( A \) be the region of the cylinder bounded by \( C_1 \) and \( C_2 \), oriented via the outward pointing normals. Thus \( \partial A = C_1 - C_2 \). Hence

\[
\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{\partial A} \mathbf{F} \cdot d\mathbf{r} = \int_A \int \left( \text{curl} \mathbf{F} \right) \cdot d\mathbf{S} = \int_A \int \left( 0, 0, 2(x - y) \right) \cdot \mathbf{n} \, dA = 0
\]

since \( \mathbf{n} \) has z-component 0. Thus \( \int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \)

2. Consider the surface \( T \) which is the intersection of the plane \( x + 2y + 3z = 1 \) with the first octant.

(a) Draw a picture of \( T \).

\[
\text{SOLUTION:}
\]

(b) Use Stokes' Theorem to evaluate \( \int_{\partial T} \mathbf{F} \cdot d\mathbf{r} \) for \( \mathbf{F} = \langle y, -2z, 4x \rangle \).

\[
\text{SOLUTION:} \text{A normal vector} \mathbf{v} \text{ to the plane is} \langle 1, 2, 3 \rangle \text{ so a unit normal vector for} \ T \text{ is} \ \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle. \ By \ Stokes', \ we \ need \ to \ evaluate}
\]

\[
\int_T \int (\text{curl} \mathbf{F}) \cdot \mathbf{n} \, dA = \int_T \int (2, -4, -1) \cdot \mathbf{n} \, dA = \int_T \int \frac{-9}{\sqrt{14}} \, dA = \frac{-9}{\sqrt{14}} \text{Area}(T).
\]
Area(T) = \frac{1}{2} |\mathbf{a} \times \mathbf{b}| \text{ where } \mathbf{a} = \langle -1, 1/2, 0 \rangle \text{ and } \mathbf{b} = \langle 0, -1/2, 1/3 \rangle, \text{ so Area}(T) = \frac{1}{2} \sqrt{\frac{7}{18}}. \text{ So } 
abla \times \mathbf{F} \cdot \mathbf{n} dA = -\frac{9}{2\sqrt{14}} \cdot \frac{\sqrt{7}}{3\sqrt{2}} = -\frac{3}{4}.

Alternative approach: Parametrize T by \( r(u, v) = (u, v, \frac{1}{3}(1 - u - 2v)) \) with domain 

\[ D = \{ 0 \leq u \leq 1 \text{ and } 0 \leq v \leq \frac{1}{2} - \frac{u}{2} \}. \]

\[ \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \end{vmatrix} = \langle 1/3, 2/3, 1 \rangle \]

So flux = \( \int_D \langle 2, -4, -1 \rangle \cdot \langle 1/3, 2/3, 1 \rangle \, dv \, du = \int_D -3 \, dA = -3 \text{ Area}(D) = -3/4. \)

3. Carefully explain how Green’s Theorem is actually a special case of Stokes’ Theorem.

**SOLUTION:**

Green’s theorem is just Stokes’ theorem for a surface and vector field that lie totally inside the xy-plane. Let’s consider the situation of Green’s theorem — we start with a vector field in the xy-plane, which looks like \( \langle P(x, y), Q(x, y) \rangle \), and a region D inside the plane with some boundary curves \( \partial D \). Let’s think about these living inside of the xy-plane in 3-space, \( \mathbb{R}^3 \), just extending in the most obvious way, so that we can see what Stokes’ theorem says. Let us now have the vector field \( \mathbf{F} \) on \( \mathbb{R}^3 \) defined by 

\[ \mathbf{F}(x, y, z) = \langle P(x, y), Q(x, y), 0 \rangle \]

The region D is now a surface which happens to lie entirely inside the plane \( z = 0 \). Computing the curl of \( \mathbf{F} \) gives 

\[ \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \langle -\frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle \]

as the functions \( P \) and \( Q \) only depend on \( x \) and \( y \). Looking at the flux of the curl of this vector field \( \mathbf{F} \) through the surface with upwards normal \( \mathbf{n} = \langle 0, 0, 1 \rangle \), we have the following expression:

Flux through \( D \) of \( \nabla \times \mathbf{F} \) = \( \int_D \nabla \times \mathbf{F} \cdot \mathbf{n} \, dS = \int_D \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle \cdot \langle 0, 0, 1 \rangle \, dS = \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dS \)

Technically, we should call the surface living inside of \( \mathbb{R}^3 \) something like \( D' \), and parametrize by \( r(x, y) = (x, y, 0) \), where the domain for \( (x, y) \) is \( D \). In this case, the surface integral above becomes exactly a standard double integral in the plane:

\[ \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA \]

On the other hand, Stokes’ theorem gives us a second expression which computes the flux of \( \nabla \times \mathbf{F} \): the line integral of \( \mathbf{F} \) along the boundary of \( D \). Note that with upward normal, the rule for the direction is exactly the same as the one for Green’s theorem – left arm points in walking
along the direction where your head points up in the direction of the normal. So, we have exactly the statement of Green's theorem:

\[
\text{Flux through } D \text{ of } \text{curl}(\mathbf{F}) = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA
\]

and

\[
\text{Flux through } D \text{ of } \text{curl}(\mathbf{F}) = \oint_{\partial D} \mathbf{F} \cdot d\mathbf{r}
\]

4. Work the following problem.

20. The magnetic field \( \mathbf{B} \) due to a small current loop (which we place at the origin) is called a **magnetic dipole** (Figure 18). Let \( \rho = (x^2 + y^2 + z^2)^{1/2} \). For \( \rho \) large, \( \mathbf{B} = \text{curl}(\mathbf{A}) \), where

\[
\mathbf{A} = \left( -\frac{y}{\rho^3}, \frac{x}{\rho^3}, 0 \right)
\]

(a) Let \( C \) be a horizontal circle of radius \( R \) with center \((0, 0, c)\), where \( c \) is large. Show that \( \mathbf{A} \) is tangent to \( C \).

(b) Use Stokes' Theorem to calculate the flux of \( \mathbf{B} \) through \( C \).

**SOLUTION:**

(a) Parametrize the circle at height \( c \) and radius \( R \) centered on the \( z \) axis by \( \mathbf{r}(t) = (R \cos(t), R \sin(t), c) \). Then, the tangent is given by

\[
\mathbf{r}'(t) = (-R \sin(t), R \cos(t), 0)
\]

Substituting \( x(t) = R \cos(t) \) and so on into the vector field \( \mathbf{A} \), we get what the vector field is at the corresponding point:

\[
\mathbf{A}(\mathbf{r}(t)) = \left( -\frac{R \sin(t)}{\rho^3}, \frac{R \cos(t)}{\rho^3}, 0 \right)
\]

It doesn't matter what \( \rho \) is exactly — we just want to know if the two vectors are multiples of each other at \( \mathbf{r}(t) \), which they are:

\[
\mathbf{r}'(t) \cdot \frac{1}{\rho^3} = \mathbf{A}(\mathbf{r}(t))
\]
(b) Let $D$ be the horizontal disk enclosed by $C$. We'll compute the flux with upward normal. By Stokes’ theorem,

$$
\iiint_D B \cdot n \, dS = \iint_D \text{curl}(A) \cdot n \, dS = \int_C A \cdot \, dr
$$

Here $C$ is traversed counterclockwise as seen from above to match the upward normal. Then, using our parametrization from before:

$$\int_C A \cdot \, dr = \int_0^{2\pi} A(r(t)) \cdot r'(t) \, dt = \int_0^{2\pi} \frac{r'(t)}{\rho^3} \cdot r'(t) \, dt = \int_0^{2\pi} \frac{R^2}{(R^2 + c^2)^{3/2}} \, dt$$

as $r'(t) \cdot r'(t) = R^2 \sin^2(t) + R^2 \cos^2(t) + 0^2$ and $\rho^3 = \left(\sqrt{(R \cos(t))^2 + (R \sin(t))^2 + c^2}\right)^3$

Because $R$ and $c$ are fixed, we get a final answer immediately of

$$\iiint_D B \cdot n \, dS = \frac{2\pi R^2}{(R^2 + c^2)^{3/2}}$$