1. Consider the region $D$ in $\mathbb{R}^3$ bounded by the $xy$-plane and the surface $x^2 + y^2 + z = 1$.

(a) Make a sketch of $D$.

**Solution.** The sketch of $D$ is shown below.

(b) The boundary of $D$, denoted $\partial D$, has two parts: the curved top $S_1$ and the flat bottom $S_2$. Parameterize $S_1$ and calculate the flux of $\mathbf{F} = (0, 0, z)$ through $S_1$ with respect to the upward pointing unit normal vector field. Check your answer with the instructor.

**Solution.** To parametrize $S_1$, one has

$$\mathbf{r}(u, v) = \langle u \cos v, u \sin v, 1 - u^2 \rangle, \quad 0 \leq u \leq 1, \ 0 \leq v \leq 2\pi.$$ 

In order to calculate the flux, first we have

$$\mathbf{r}_u = \langle \cos v, \sin v, -2u \rangle, \quad \mathbf{r}_v = \langle -u \sin v, u \cos v, 0 \rangle,$$

and so

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 2u^2 \cos v, 2u^2 \sin v, u \rangle.$$

Therefore, the flux of $\mathbf{F} = (0, 0, z)$ through $S_1$ with respect to the upward pointing unit normal vector field is

$$\iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS = \int_0^{2\pi} \int_0^1 \mathbf{F}(u, v) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, du \, dv$$

$$= \int_0^{2\pi} \int_0^1 (0, 0, 1 - u^2) \cdot (2u^2 \cos v, 2u^2 \sin v, u) \, du \, dv$$

$$= \int_0^{2\pi} \int_0^1 (1 - u^2) u \, du \, dv = \frac{\pi}{2}.$$
(c) Without doing the full calculation, determine the flux of $\mathbf{F}$ through $S_2$ with the downward pointing normals.

**Solution.** Since $\mathbf{F} = 0$ on $S_2$, we know the flux of $\mathbf{F}$ through $S_2$ is

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_2} 0 \cdot \mathbf{n} \, dS = 0.$$  

(d) Determine the flux of $\mathbf{F}$ through $\partial D$ with the outward pointing normals.

**Solution.** By adding up the result from (a) and (b), one gets

$$\iint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, dS = \iint_{S_1} \mathbf{F} \cdot \mathbf{n} \, dS + \iint_{S_2} \mathbf{F} \cdot \mathbf{n} \, dS = \frac{\pi}{2}.$$  

(e) Apply the Divergence Theorem and your answer in (d) to find the volume of $D$. Check your answer with the instructor.

**Solution.** By the Divergence Theorem, one has

$$\iiint_D \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_D \text{div} \mathbf{F} \, dV.$$  

Since $\text{div} \mathbf{F} = 1$, one gets

$$\text{Volume}(D) = \iiint_D 1 \, dV = \iiint_D \text{div} \mathbf{F} \, dV = \iiint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, dS = \frac{\pi}{2}.$$  

2. Consider the vector field $\mathbf{F} = (-y, x, z)$.

(a) Compute curl $\mathbf{F}$.

**Solution.**

$$\text{curl} \, \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & z \end{vmatrix} = (0, 0, 2).$$  

(b) For the surface $S_1$ above, evaluate $\iint_{S_1} (\text{curl} \, \mathbf{F}) \cdot \mathbf{n} \, dA$.

**Solution.** By applying the parametrization of $S_1$ in 1(b), one gets

$$\iint_{S_1} (\text{curl} \, \mathbf{F}) \cdot \mathbf{n} \, dA = \int_0^{2\pi} \int_0^1 (\text{curl} \, \mathbf{F}) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dudv = \int_0^{2\pi} \int_0^1 2u \, dudv = 2\pi.$$
(c) Check your answer in (b) using Stokes’ Theorem.

**Solution.** The boundary of $S_1$ is a unit circle centered at the origin in the $xy$-plane. So we can parametrize it as

\[ C : \mathbf{r}(t) = (\cos t, \sin t, 0), \quad 0 \leq t \leq 2\pi. \]

Thus, by Stokes’ Theorem, one has

\[
\iint_{S_1} \left( \text{curl} \, \mathbf{F} \right) \cdot \mathbf{n} \, dA = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt
\]

\[
= \int_0^{2\pi} (-\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0) \, dt = \int_0^{2\pi} 1 \, dt = 2\pi.
\]

3. If time remains:

(a) Check your answer in 1(e) by directly calculating the volume of $D$.

**Solution.** One can use the polar coordinate to calculate the volume of $D$ in 1(e). Let

\[ x = r \cos \theta, \quad y = r \sin \theta, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi. \]

Then

\[
\text{Volume}(D) = \iiint_{x^2 + y^2 \leq 1} (1 - x^2 - y^2) \, dA = \int_0^{2\pi} \int_0^1 (1 - r^2) \, r \, dr \, d\theta = \frac{\pi}{2}.
\]

(b) Repeat 2 (b-c) for the surface $S_2$ and also for the surface $\partial D$. What exactly does 2(c) mean for the surface $\partial D$?

**Solution.** The normal vector of $S_2$ pointing downward is $\mathbf{n} = -\mathbf{k}$. Thus,

\[
\iint_{S_2} \left( \text{curl} \, \mathbf{F} \right) \cdot \mathbf{n} \, dA = \iiint_{S_2} (0, 0, 2) \cdot (0, 0, -1) \, dA = -2 \iint_{S_2} dA = -2\pi.
\]

To check the above answer using Stokes’ Theorem, one needs the parametrization of the boundary of $S_2$. Notice that this boundary is the same as that of $S_1$ except the orientation. The boundary of $S_2$ is parametrized by

\[ C' : \mathbf{r}(t) = (\cos(2\pi - t), \sin(2\pi - t), 0), \quad 0 \leq t \leq 2\pi. \]

Thus, by Stokes’ Theorem, one has

\[
\iint_{S_2} \left( \text{curl} \, \mathbf{F} \right) \cdot \mathbf{n} \, dA = \int_{C'} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt
\]

\[
= \int_0^{2\pi} (-\sin(2\pi - t), \cos(2\pi - t), 0) \cdot (\sin(2\pi - t), -\cos(2\pi - t), 0) \, dt
\]

\[
= \int_0^{2\pi} -1 \, dt = -2\pi.
\]
By adding up the two integrals one gets
\[
\iint_{\partial D} (\text{curl } F) \cdot \mathbf{n} \, dA = \iint_{S_1} (\text{curl } F) \cdot \mathbf{n} \, dA + \iint_{S_2} (\text{curl } F) \cdot \mathbf{n} \, dA = 2\pi + (-2\pi) = 0.
\]

Since \(\partial D\) is a surface without any curve boundary, then 2(c) shows that the integral of \(F\) along the curve boundary of the surface \(\partial D\) must be 0.

(c) For the vector field \(F = (-y, x, z)\) from the second problem, compute \(\text{div(curl } F)\). Now suppose \(F = (F_1, F_2, F_3)\) is an arbitrary vector field. Can you say anything about the function \(\text{div(curl } F)\)?

**Solution.** We already know, in 2(a), that \(\text{curl } F = (0, 0, 2)\), so \(\text{div(curl } F) = 0\). Generally, suppose suppose \(F = (F_1, F_2, F_3)\) is an arbitrary vector field. Then

\[
\text{curl } F = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
F_1 & F_2 & F_3
\end{vmatrix}
= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right),
\]

and

\[
\text{div(curl } F) = \frac{\partial}{\partial x} \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) + \frac{\partial}{\partial y} \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) + \frac{\partial}{\partial z} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)
= \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} + \frac{\partial^2 F_1}{\partial y \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0.
\]