Green's Theorem

Green's Theorem is a 2-dimensional version of the Fundamental Theorem of Calculus: it relates the (integral of) a vector field $\mathbf{F}$ on the boundary of a region $D$ to the integral of a suitable derivative of $\mathbf{F}$ over the whole of $D$.

1. Let $D$ be the unit square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$ and consider the vector field $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y) \rangle = \langle xy, x + y \rangle$. See below right for a plot.
   (a) For the curve $C = \partial D$ oriented counterclockwise, directly evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$. Hint: to speed things up, have each group member focus on one side of $C$.
   (b) Now compute $\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA$.
   (c) Check that Green's Theorem works in this example.

2. Compute the line integral of $\mathbf{F}(x, y) = \langle x^3, 4x \rangle$ along the path $C$ shown at right against a grid of unit-sized squares. To save work, use Green's Theorem to relate this to a line integral over the vertical path joining $B$ to $A$. Hint: Look at the region $D$ bounded by these two paths. Check your answer with the instructor.

3. Consider the quarter circle $C$ shown below and the vector field $\mathbf{F}(x, y) = \langle 2xe^y, x + x^2e^y \rangle$. The goal of this problem is to compute the line integral $I_0 = \int_C \mathbf{F} \cdot d\mathbf{r}$. 

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**Plotted Figures**

- Vector field $\mathbf{F}$ on a unit square $D$.
- Line integral along path $C$.
- Quarter circle $C$.
(a) Parameterize $C$ and start directly expanding out $I_0$ into an ordinary integral in $t$ until you are convinced that finding $I_0$ this way will be a highly unpleasant experience.

(b) Check that $F$ is not conservative, so we can’t use that trick directly to compute $I_0$.

(c) Find a function $f(x, y)$ such that $F = G + \nabla f$, where $G$ is the vector field $\langle 0, x \rangle$.

(d) Argue geometrically that $G$ integrates to 0 along any line segment contained in either the $x$-axis or the $y$-axis.

(e) Use part (d) with Green’s Theorem to show that $\int_C G \cdot dr = 4\pi$.

(f) Combine parts (c–e) with the Fundamental Theorem of Line Integrals to evaluate $I_0$. Check your answer with the instructor.

4. Consider the shaded region $V$ shown, bounded by a circle $C_1$ of radius 5 and two smaller circles $C_2$ and $C_3$ of radius 1. Suppose $F(x, y) = \langle P, Q \rangle$ is a vector field where $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2$ on $V$. Assuming in addition that $\int_{C_2} F \cdot dr = 3\pi$ and $\int_{C_3} F \cdot dr = 4\pi$, compute $\int_{C_1} F \cdot dr$. Check your answer with the instructor.

5. Suppose $D$ is a region in the plane bounded by a closed curve $C$. Use Green’s Theorem to show that both $\int_C x \, dy$ and $-\int_C y \, dx$ are equal to $\text{Area}(D)$.

6. The curve satisfying $x^3 + y^3 = 3xy$ is called the Folium of Descartes and is shown at right.

(a) Let $C$ be the “bulb” part of this folium, more precisely, the part in the positive quadrant. Show that any line $y = tx$ for $t > 0$ meets $C$ in exactly two points, one of which is the origin. Use this fact to parameterize $C$ by taking the slope $t$ as the parameter.

(b) Use part (a) and Problem 5 to compute the area bounded by $C$. Check your answer with the instructor.