Lecture 37: More on Stokes' Theorem (§16.8), including understanding the curl (§16.5) and the connection to conservative vector fields (§16.8).

Last time: Stokes' Thm: $S$ surface in $\mathbb{R}^3$
$\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ vector field. Then
\[ \int_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\text{curl} \, \vec{F}) \cdot \vec{n} \, dA \]

Ex: $\vec{F} = (-y, x, yz)$
$\text{curl} \, \vec{F} = (z, 0, 2)$

\[ \iint_S (\text{curl} \, \vec{F}) \cdot \vec{n} \, dA = 2\pi = \int_C \vec{F} \cdot d\vec{r} \text{ for all of these!} \]

[Takes some getting used to, is really just Green's Thm/2d Divergence Thm in disguise...]

Check the easy one: $\iint_D (\text{curl} \, \vec{F}) \cdot \vec{n} \, dA$

\[ = \iint_D (z, 0, 2) \cdot (0, 0, 1) \, dA = \iint_D 2 \, dA = 2 \text{Area}(O) = 2\pi \]
Note: Also works when \( S \) has several boundary components, \([\text{provided they are oriented correctly}]\)

\[ \vec{F} = \text{fluid flow} \]

Q: How fast does it spin?

A: \[ \omega = \frac{1}{2\pi r^2} \oint_{C_r} \vec{F} \cdot d\vec{r} \]

units = \( \frac{\text{radians}}{\text{time}} \)

Plausible:
\( a \) Want tangential component of \( \vec{F} \) as it hits the paddles.
\( b \) Looks like an average (almost).

Stokes says:
\[ \omega = \frac{1}{2\pi r^2} \iint_{D_r} (\text{curl } \vec{F}) \cdot \hat{n} \ dA \]
\[ = \frac{1}{2} \left( \frac{1}{{\text{Area}(D_r)}} \int_{D_r} (\text{curl} \, \vec{F}) \cdot \vec{n} \, dA \right) \]

Taking \( r \to 0 \) get: \( \omega = \frac{1}{2} (\text{curl} \, \vec{F}) \cdot \vec{n} \)

So the rate of rotation is fastest the direction of \( \text{curl} \, \vec{F}(\rho) \) and then \( \omega = \frac{1}{2} |\text{curl} \, \vec{F}| \).

Note: A vector field where \( \text{curl} \, \vec{F} = \vec{0} \) everywhere are called \underline{irrotational}.

Ex: \( \vec{F} = \frac{1}{x^2 + y^2} (-y, x, 0) \) has \( \text{curl} \, \vec{F} = \vec{0} \) except at \((0,0)\) where it's not defined.

Experimentally, a draining tub is an \underline{irrotational flow}!
Conservative Vector Fields. Recall $\vec{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is conservative if $\vec{F} = \nabla f$ for some $f : \mathbb{R}^n \rightarrow \mathbb{R}$.

**Thm B:** Suppose $\vec{F} = (P, Q)$ is a vector field on an open simply-connected region $D$ in $\mathbb{R}^2$. Then $\vec{F}$ is conservative if and only if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$ on $D$.

[Want this kind of easy to use test for $\mathbb{R}^3$.]

**Thm B':** A vector field $\vec{F}$ defined on all of $\mathbb{R}^3$ is conservative if and only if $\text{curl} \vec{F} = \vec{0}$ everywhere.

**Note:** Example on last page has $\text{curl} \vec{F} = \vec{0}$ but is not conservative because of:

**Thm A:** A vector field $\vec{F}$ on an open connected region $R$ in $\mathbb{R}^n$ is conservative if and only if

$$\int_C \vec{F} \cdot d\vec{r} = 0 \text{ for every closed curve } C \text{ in } R.$$

Compatible with Thm B' since $\vec{F}$ is not defined at $(0, 0, 0)$. 
Idea behind Thm B', part I. Suppose \( \vec{F} \) is conservative, with \( \vec{F} = \nabla f \) for some \( f: \mathbb{R}^3 \to \mathbb{R} \). Then

\[
\text{curl } \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix} = \left( \frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, 0, 0 \right)
\]

and so \( \text{curl } \vec{F} = \hat{\imath} \) everywhere.

Next time: I'll explain why \( \text{curl } \vec{F} = \hat{\imath} \) means \( \vec{F} \) is conservative.