Lecture 36: Stokes Theorem (§16.8), including the definition of the curl (§16.5)

Last time: \( S \) surface in \( \mathbb{R}^3 \), \( \vec{F} : \mathbb{R}^3 \to \mathbb{R}^3 \) a vector field

Flux = \[ \iint_S (\vec{F} \cdot \vec{n}) \, dA \] where \( \vec{n} \) is a unit normal vector field.

Divergence Thm: \( D \) a region in \( \mathbb{R}^3 \), \( \vec{F} : D \to \mathbb{R}^3 \) a vector field. Then

\[ \iiint_D \text{div} \, \vec{F} \, dV = \iint_S (\vec{F} \cdot \vec{n}) \, dA \]

Integral Thms:

1. \( 1-d \): \( a \to b \)
\[ f(b) - f(a) = \int_a^b f'(t) \, dt \]

2. \( 2-d \): Green's Thm
\( \vec{F} : \mathbb{R}^2 \to \mathbb{R}^2 \)
\[ \int_C \vec{F} \cdot d\vec{r} = \iint_D \left( \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x} \right) \, dA \]
where \( \vec{F} = (P, Q) \).

3. \( 3-d \): Divergence Thm.
Curl: \( \vec{F} : \mathbb{R}^3 \to \mathbb{R}^3 \) a vector field with \( \vec{F} = (F_1, F_2, F_3) \)

\[
\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}
\]

Another vector field.

\[
= \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \hat{i} - \left( \frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \hat{j} + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \hat{k}
\]

Ex: \( \vec{F} = (y, 0, 0) \)

\[\text{curl } \vec{F} = (0, 0, -1)\]

Q: What does curl measure?

First, consider \( \vec{F} = (F_1, F_2, 0) \). Which is effectively a vector field on \( \mathbb{R}^2 \). Place a small paddle wheel into the flow. As it moves along with the flow, \(| \text{curl } \vec{F} | \) is the rate of rotation [precisely \(2\) (angular velocity)].

\[\text{curl } \vec{F} = (0, 0, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})\] where \( \omega > 0 \) if rotation is anticlockwise.
Stokes Thm: $S$ surface in $\mathbb{R}^3$ with boundary curve $C$.

$\vec{F}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ a vector field.

\[ \int_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl}\vec{F}) \cdot \vec{n} \, dA \]

Relation to Green's Thm:

$\vec{F} = (F_x, F_y, 0)$

$\vec{n} = \hat{k}$

So

$$(\text{curl}\vec{F}) \cdot \vec{n} = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}$$

$$= \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Correlated so that $S$ is to your left as you walk around $C$ with head pointing in the normal direction.

Also, $S$ is orientable.
Example: \( S = \) upper unit hemisphere
\( \hat{n} = \) outward normal
\( \vec{F}( -y, x, yz ) \)

\[
\text{curl} \vec{F} = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-1 & 1 & z
\end{vmatrix} = (z, 0, 2)
\]

\[
\iint_S (\text{curl} \vec{F}) \cdot \hat{n} \ dA = \iiint_S (z, 0, 2) \cdot (x, y, z) \ dA
\]

\[
= \iint_S xz + 2z \ dA = \iint_S 2z \ dA
\]

integrates to 0 by symmetry.

Parameterize \( S \):

\[
\vec{r}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)
\]

\( dA = \sin \phi \ d\theta \ d\phi \)

\[
\iint_S 2z \ dA = \int_0^{2\pi} \int_0^{\pi/2} 2 \cos \phi \sin \phi \ d\phi \ d\theta
\]

\[
= \int_0^{2\pi} \sin^2 \phi \Big|_{\phi = 0}^{\pi/2} \ d\theta = \int_0^{2\pi} 1 \ d\theta = 2\pi
\]
\[
\text{Compare: } \mathbf{f}(t) = (\cos t, \sin t, 0)
\]
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{f}(t)) \cdot \mathbf{f}'(t) \, dt
\]
\[
= \int_0^{2\pi} (-\sin t, \cos t, 0) \cdot (-\sin t, \cos t, 0) \, dt
\]
\[
= \int_0^{2\pi} \sin^2 t + \cos^2 t \, dt = \int_0^{2\pi} 1 \, dt = \left. 2\pi \right|_0^{2\pi}
\]
So Stokes' Theorem works in this case!

What about the lower hemisphere?
\[
\iiint_{S'} (\text{curl } \mathbf{F}) \cdot \mathbf{n} \, dA = \cdots = \int_{\phi = \pi}^{\phi = \pi/2} \int_0^{2\pi} \sin^2 \phi \, \sin \theta \, d\theta \, d\phi
\]
\[
= \int_0^{2\pi} -1 \, d\theta = -2\pi.
\]

But: C is also the boundary of S', so shouldn't we get \( \int_C \mathbf{F} \cdot d\mathbf{r} = 2\pi \)?

Solution: With outward normal, C gets oriented the other way by S'.