Last time: Flux
\[ \vec{F} : \mathbb{R}^2 \to \mathbb{R}^2 \]

Flux = rate water is crossing \( C \)
\[ \int_C (\vec{F} \cdot \vec{n}) \, ds \]
where \( \vec{n} \) is a unit normal vector field along \( C \).

Today, will use the concept of flux to give a different formulation of Green's Thm., one that will explain why it works, and will also generalize to \( \mathbb{R}^3 \).

**Divergence:** \( \vec{F} : \mathbb{R}^2 \to \mathbb{R}^2 \) a vector field with \( \vec{F} = (F_1, F_2) \)
\[ \text{div} \, \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \]
which is \([Q.7] \) a function \( \mathbb{R}^2 \to \mathbb{R} \)
\[ \nabla \cdot \vec{F} \]

**Meaning:** [Thinking of \( \vec{F} \) as rep. fluid flow: ]
\[ \text{div} \, \vec{F} = \text{rate of expansion of area under the flow.} \]

**Ex:** \( \vec{F} = (x, y) \)

Suppose we dye the water inside the unit circle green.
Reason: Consider the flow on the x-axis: 
\[ x(t) = x(t) \Rightarrow x(t) = e^t \]

In general, the follow of \((x_0, y_0)\) is given by \( \dot{\mathbf{r}}(t) = (x_0 e^t, y_0 e^t) \) since \( \dot{\mathbf{r}}(t) = \dot{\mathbf{r}}(t) = \mathbf{F}(\mathbf{r}(t)) \)

So
\[
\frac{\text{Green Area}(t)}{\text{Green Area @ } t=0} = \frac{2\pi (e^t)^2}{2\pi} = e^{2t} \text{ in the green area.}
\]

Note
\[ \text{div} \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = 2. \]

In real life, how could this happen?

- Fluid becomes more/less dense (compressible)
  [When modeling airflow over a wing, usually assume \( \text{div} \mathbf{F} = 0 \) if speeds are < Mach 0.3]
- Fluid is being added somehow.
Approx Picture:

\[ \Delta t \vec{F}(x_0, y_0) \]

\[(x_0, y_0) \quad \Delta t \vec{F}(x_0, y_0)\]

\[\vec{v} \approx (\Delta x, 0) + \Delta t \Delta x \vec{F}_x(x_0, y_0) = (\Delta x + \Delta t \Delta x \frac{\partial F_1}{\partial x}, \Delta t \Delta x \frac{\partial F_2}{\partial x})\]

\[= \Delta x \left(1 + \Delta t \frac{\partial F_1}{\partial x}, \Delta t \frac{\partial F_2}{\partial x}\right)\]

\[\vec{w} \approx \Delta y \left(\Delta t \frac{\partial F_1}{\partial y}, 1 + \Delta t \frac{\partial F_2}{\partial y}\right)\]

So the new area is

\[= \Delta x \Delta y \left(1 + \Delta t (\text{div } \vec{F}) + \Delta t^2 (\text{stuff})\right)\]

Thus:

\[\frac{\text{new area}}{\text{old area}} \approx 1 + \Delta t (\text{div } \vec{F})\]

and so the rate of expansion is \((\text{div } \vec{F})\).
Check units:
\(\overline{F}(x,y)\) has units \(\text{m/s}\)
\(\text{div}\overline{F}\) has units \(\text{m/s}^{-1}\) and so \(1+\Delta t\text{ (div}\overline{F}\text{)}\)
is dimensionless.

Divergence Thm: \(D\) a region in \(\mathbb{R}^2\) bounded by \(C\), with outward unit normals \(\hat{n}\).
Then
\[
\int_C \overline{F} \cdot \hat{n} \, ds = \iint_D \text{div}\overline{F} \, dA
\]

Flux

Reason: After time \(\Delta t\), the fluid in \(D\) now fills the region \(D'\). Now

\[
\frac{\text{Area}(D')}{\text{Area}(D)} \approx 1 + \Delta t \cdot r
\]

Where \(r = \text{Ave. rate of expansion} = \frac{1}{\text{Area}(D)} \iint_D \text{div}\overline{F} \, dA\)

\[\Rightarrow\]
\[
\text{Area}(D') - \text{Area}(D) \approx \Delta t \iint_D \text{div}\overline{F} \, dA
\]
Now as the region $D$ is fixed, the change in area can only be accomplished by fluid crossing $C$. The amount that crosses $C$ in time $\Delta t$ is $\approx \Delta t \int_C (\vec{F} \cdot \vec{n}) \, ds$. Thus we must have $\int_C \vec{F} \cdot \vec{n} \, ds = \iint_D \text{div} \vec{F} \, dA$

How this relates to Green’s Theorem:

If $\vec{F} = (F_1, F_2)$ set $\vec{G} = (-F_2, F_1) = (P, Q)$

Then

\[ \int_C (\vec{F} \cdot \vec{n}) \, ds = \int_C \vec{G} \cdot d\vec{r} \quad \bigstar \]

and

\[ \text{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \]

so

\[ \iint_D \text{div} \vec{F} \, dA = \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \, dA \]
Reason for \(\ast\): Take a unit-speed param \(\vec{r}: [a, b] \rightarrow \mathbb{R}^3\).

Get \(\vec{n}(t)\) from \(\vec{r}'(t)\) by rotating right:

\[
T(u, v) = (v, -u)
\]

So \(\vec{n}(t) = (r_2'(t), -r_1'(t))\) and

\[
\int_C \vec{F} \cdot \vec{n} \, ds = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{n}(t) \, dt
\]

\[
= \int_a^b F_1(\vec{r}(t)) r_2'(t) - F_2(\vec{r}(t)) r_1'(t) \, dt
\]

\[
= \int_a^b G(\vec{r}(t)) \cdot \vec{r}'(t) \, dt
\]

\[
= \int_C G \cdot d\vec{r}
\]

Next time: Flux across a surface.