Lecture 30: Area and integration on surfaces (§16.6 and §16.7)

Last time: Parameterizing surfaces

\[ \mathbf{\hat{r}} : (D \text{ in } \mathbb{R}^2) \rightarrow (S \text{ in } \mathbb{R}^3) \]

Q: How do we compute the area of $S$?

Easy:

\[ \Delta x_0 \]

Hard:

Point: Really, the same idea as change of coordinates, basic line sets, etc...

Idea:
We approximate the area of a region at right using a parallelogram:

So the area is

\[ \approx \left| (\Delta u \vec{F}_u) \times (\Delta v \vec{F}_v) \right| \]

\[ = |\vec{F}_u \times \vec{F}_v| \Delta u \Delta v \]

So,

\[ \text{Area}(S) \approx \sum_{\text{small \ } \text{necks}} |\vec{F}_u \times \vec{F}_v| \Delta u \Delta v. \]

Let \( \Delta u, \Delta v \to 0 \), then

\[ \text{Area}(S) = \iint_D |\vec{F}_u \times \vec{F}_v| \, du \, dv \]

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Ex: Unit sphere

\[ \vec{F}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) \]
\[
\vec{r}_\phi \times \vec{r}_\theta = 
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\cos \phi \cos \theta & \cos \phi \sin \theta & -\sin \phi \\
-\sin \phi \sin \theta & \sin \phi \cos \theta & 0
\end{vmatrix}
\]

\[
= \left( +\sin^2 \phi \cos \theta, -\sin^2 \phi \sin \theta, \sin \phi \cos \phi (\cos^2 \theta + \sin^2 \theta) \right)
\]

\[
|\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{\sin^4 \phi + \sin^2 \phi \cos^2 \phi} = \sqrt{\sin^2 \phi} = \sin \phi
\]

Geometrically, we saw this \( \sin \phi \) before when we were discussing spherical coordinates.

\[
\text{Area } (\Theta) = \iint_D \sin \phi \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} \int_0^\pi \sin \phi \, d\phi \, d\theta
\]

\[
= \int_0^{2\pi} \left[ -\cos \phi \right]_{\phi=0}^{\phi=\pi} \, d\theta
\]

\[
= \left[ -(-1) - (-1) \right] = 2
\]

\[
= \int_0^{2\pi} 2 \, d\theta = \frac{4\pi}{\pi}
\]
Note: \[
\text{Area } 2\{ 1 \} = 2 \text{Area } (\bigcirc) + \text{Area } (\text{\textcircled{1}})
\]
\[
= 2\pi + 4\pi = 6\pi.
\]

Hence
\[
\frac{\text{Area } (\text{\textcircled{1}})}{\text{Area } (\bigcirc)} = \frac{3}{2} = \frac{\text{Vol } (\text{\textcircled{1}})}{\text{Vol } (\bigcirc)}
\]

Isn't typical, e.g. compare a cake with a sphere.

With curves finding \[
\int_C \text{Length } = \int_C ds = \int_a^b |\vec{r}'(t)| dt \quad \text{where } \vec{r} : [a,b] \rightarrow \mathbb{C}.
\]

lead to integrating \(f: \mathbb{R}^3 \rightarrow \mathbb{R}\) along \(C\) via:
\[
\int_C f ds = \int_a^b f(\vec{r}(t)) |\vec{r}'(t)| dt
\]

Similarly, if \(f: \mathbb{R}^3 \rightarrow \mathbb{R}\) and \(S\) is a surface,
\[
\iint_S f \, dA = \iint_D f(\vec{r}(u,v)) |\vec{r}_u \times \vec{r}_v| \, du \, dv
\]

where \(\vec{r}: D \rightarrow S\) is a parameterization.
Ex: Find the average of $f(x, y, z) = xy + z$ on the cone $S$ given by $x^2 + y^2 = z^2$ with $0 \leq z \leq 1$.

Parameters: height angle

$\mathbf{r}(u, v) = (v \cos u, v \sin u, v)$

Area: $\mathbf{r}_u = (-v \sin u, v \cos u, 0)$

$\mathbf{r}_v = (\cos u, \sin u, 1)$

$|\mathbf{r}_u \times \mathbf{r}_v| = |(v \cos u, v \sin u, -v)| = \sqrt{2}v$

So,

$\text{Area} = \iint_D 1 \, dA = \iint_D \sqrt{2} v \, du \, dv = \int_0^1 \int_0^{2\pi} \sqrt{2} v \, du \, dv$

$= \int_0^1 2\sqrt{2} \pi v \, dv = \sqrt{2}\pi$
\[
\text{Average} = \frac{1}{\text{Area}} \iint_S xy + z \, dA \\
= \frac{1}{\sqrt{2}\pi} \int_0^1 \int_0^{2\pi} \left( v^2 \sin u \cos u + v \right) \left( \sqrt{2} \, v \right) \, du \, dv \\
= \frac{1}{\pi} \int_0^1 \left( \int_0^{2\pi} v^3 \sin u \cos u + v^2 \, du \right) \, dv \\
= \frac{1}{\pi} \int_0^1 \left( \frac{1}{2} v^3 \sin^2 u + v^2 \bigg|_{u=0}^{u=2\pi} \right) \, dv \\
= \frac{1}{\pi} \int_0^1 2\pi v^2 \, dv = 2 \int_0^1 v^2 \, dv = 2 \frac{v^3}{3} \bigg|_{v=0}^{1} = \frac{2}{3}.
\]