Lecture 29: Surfaces in $\mathbb{R}^3$ (§16.6)

In general, for $T : \mathbb{R}^3 \to \mathbb{R}^3$

$$\iiint_S f(T(u,v,w)) \left| \det J \right| \, du \, dv \, dw = \iiint_{T^{-1}(S)} f(x,y,z) \, dx \, dy \, dz$$

where

$$J = \begin{vmatrix} \frac{\partial T_1}{\partial u} & \frac{\partial T_1}{\partial v} & \frac{\partial T_1}{\partial w} \\ \frac{\partial T_2}{\partial u} & \frac{\partial T_2}{\partial v} & \frac{\partial T_2}{\partial w} \\ \frac{\partial T_3}{\partial u} & \frac{\partial T_3}{\partial v} & \frac{\partial T_3}{\partial w} \end{vmatrix}$$

if $T = (T_1, T_2, T_3)$.

Surfaces in $\mathbb{R}^3$: [Things that look locally like $\mathbb{R}^2$]

- Plane
- Sphere
- Graph of $f : \mathbb{R}^2 \to \mathbb{R}$
Parameterization:

Curves:
\[ \mathbf{r} : [a, b] \rightarrow \mathbb{R}^3 \]

Surface:
\[ \mathbf{F} : D \rightarrow \mathbb{R}^3 \]

Ex: \( S = \text{graph of } f(x,y) = -x^2 - y^2 \)

\[ \mathbf{r} : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \text{ given by } \mathbf{r}(u,v) = (u,v,-u^2-v^2) \]

Note: \( \mathbf{r}(u,v) = (u,v,f(u,v)) \) works to parameterize the graph of any \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \).
Ex: \( S = \) cylinder where \( y^2 + z^2 = 1 \)

To specify a point, need to give:

- \( x \) coord \( \rightarrow u \)
- \( \theta \) angle \( \rightarrow v \)

\( D = \{ 0 \leq v \leq 2\pi \} \)

\( \bar{F}: D \rightarrow \mathbb{R}^3 \)

\( \bar{F}(u, v) = (u, \cos v, \sin v) \)

Ex: \( \theta \)

\( D \)

\( F(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi) \)

Many more examples:
Tangent planes: [Encountered before, for graphs and level sets...]
Locally approximate $S$.
[Compare $\nabla f$.]
Consider a parametric $\vec{r}: D \rightarrow \mathbb{R}^3$

Consider the curve
$$\vec{F}(t) = \vec{r}(u_0 + t, v_0)$$
which passes through $\vec{x}$ at time $t = 0$. Its tangent vector at $t = 0$ is
$$\vec{F}'(0) = \left. \frac{d}{dt} \right|_{t=0} (r_1(u_0 + t, v_0), r_2(u_0 + t, v_0), r_3(u_0 + t, v_0))$$
$$= \left( \frac{\partial r_1}{\partial u}(u_0, v_0), \frac{\partial r_2}{\partial u}(u_0, v_0), \frac{\partial r_3}{\partial u}(u_0, v_0) \right)$$
$$= \vec{r}_u(u_0, v_0)$$

where $\vec{r}(u, v) = (r_1(u, v), r_2(u, v), r_3(u, v))$
Similarly we have \( \vec{r}_v(u_0, v_0) \)

Together, \( \vec{r}_u \) and \( \vec{r}_v \) span the tangent plane to \( S \) at \( \vec{r} \).

**Ex:** Find the tangent plane to the unit sphere at \((\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})\). In terms of

\[
\vec{r}(\phi, \theta) = (\sin \phi \cos \theta, \sin \phi \sin \theta, \cos \phi)
\]

This is \( \vec{r}(\pi/4, 0) \).

\[
\vec{r}_\phi = (\cos \phi \cos \theta, \cos \phi \sin \theta, -\sin \phi)
\]

\[
\vec{r}_\phi(\pi/4, 0) = (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})
\]

\[
\vec{r}_\theta = (-\sin \phi \sin \theta, \sin \phi \cos \theta, 0)
\]

\[
\vec{r}_\theta(\pi/4, 0) = (0, \frac{1}{\sqrt{2}}, 0)
\]
So a normal vector is
\[ \mathbf{n} = \mathbf{r}_\phi \times \mathbf{r}_\theta \]
\[ = \left( \frac{1}{2}, 0, \frac{1}{2} \right) \]
which points straight out from the sphere, as we expect.