Lecture 27: Changing coordinates I (§15.9)

Previously: \( \theta \)

\[ T: \mathbb{R}^2 \to \mathbb{R}^2 \quad T(r, \theta) = (r \cos \theta, r \sin \theta) \]

\[
\iint_R f(x, y) \, dA = \iint_S f(T(r, \theta)) \, rdrd\theta
\]

Next two lectures: general change of coordinates...

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Suppose we want to integrate many functions over the region shown at right. For each, would need two integrals to do so:

\[
\iint_R f(x, y) \, dA = \int_0^2 \int_{y = \frac{x}{2}}^{x} f(x, y) \, dy \, dx + \int_2^3 \int_{y = 2x - 3}^{x} f(x, y) \, dy \, dx.
\]

Goal: Do a change of coordinates so that can use just one integral. Sing praises thereof.
Need $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ so that $T(S) = \mathbb{R}$.

Simpliest kind of $T$: Linear Transformations.

$$T_A(u, v) = (au + bv, cu + dv) \text{ for some } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Ex: $T(u, v) = (u - 2v, u + 2v)$, $A = \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}$

from Thursday's worksheet.

Key properties:

1. $T(\text{Line}) = \text{Line}$
2. $T(0, 0) = (0, 0)$
3. $T$ det. by $T(1, 0) = (a, c)$
   and $T(0, 1) = (b, d)$
4. $\overrightarrow{w_1}, \overrightarrow{w_2}$ in $\mathbb{R}^2$, $s, t \in \mathbb{R}$
   $$T(s\overrightarrow{w_1} + t\overrightarrow{w_2}) = sT(\overrightarrow{w_1}) + tT(\overrightarrow{w_2})$$
If we want $T(5) = \mathbb{R}$
then $T(1,0) = (2,1)$ and $T(0,1) = (3,3)$.

Hence $A = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ and $T(u,v) = (2u+3v, u+3v)$
if $T$ is linear, and that will work since such send lines to lines.

To integrate, need to understand how $T$ distorts area.

$\text{Area}(T(\square)) = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3$

Now a linear transform distorts area uniformly.
(think about problem 1 on the worksheet)
Hence \[ \text{Area (\square)} = \Delta u \Delta v \quad \rightarrow \quad \text{Area (\triangle)} = 3 \Delta u \Delta v \]

So: \[ dA = 3 \, du \, dv \]

Example: \[ \iint_R x - y \, dA = \iint_S (2u+3v)-(u+3v) \, 3 \, du \, dv \]

\[ T(u,v) = (2u+3v, u+3v) = (x,y) \]

\[ = \int_0^1 \int_0^{1-u} \quad u \quad 3 \, dv \, du = \int_0^1 3u(1-u) \, du \]

\[ = \int_0^1 3u - 3u^2 \, du = \frac{3}{2} u^2 - u^3 \bigg|_{u=0}^{u=1} = \frac{1}{2}. \]

Further: Do as shown on pg 89.
In general, if $T: \mathbb{R}^2 \to \mathbb{R}^2$ is linear, with $T(a, b) = (au+bu, cu+dv)$ for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

\[ dA = \left| \begin{array}{cc} a & b \\ c & d \end{array} \right| \, du \, dv. \]

Thus

\[ \iint_R f(x, y) \, dA = \iint_S f(T(u, v)) \left| \begin{array}{cc} a & b \\ c & d \end{array} \right| \, du \, dv \]

Linear Approx: Recall if $g: \mathbb{R}^2 \to \mathbb{R}$ is diff at $(u, v)$ then

\[ g(u+\Delta u, v+\Delta v) = g(u, v) + g_u(u, v) \Delta u + g_v(u, v) \Delta v + E(\Delta u, \Delta v) \]

Now consider $T: \mathbb{R}^2 \to \mathbb{R}^2$

\[ T(u, v) = (g(u, v), h(u, v)) \].
Then

\[ T(u+\Delta u, v+\Delta v) = T(u, v) + J_{(u,v)} (\Delta u, \Delta v) + \text{Error} \]

where

\[ J_{(u,v)} \]

is the linear transformation with

matrix \[
\begin{pmatrix}
g_u(u,v) & g_v(u,v) \\ h_u(u,v) & h_v(u,v)
\end{pmatrix}
\]

Reason

\[ T(u+\Delta u, v+\Delta v) \]

\[ \approx \left( g(u,v) + g_u(u,v) \Delta u + g_v(u,v) \Delta v, \right. \]
\[ \left. h(u,v) + h_u(u,v) \Delta u + h_v(u,v) \Delta v \right) \]

Thus locally, \( T \) looks like a linear transformation...

--- to be continued ---