Lecture 19: More on integrating vector fields (§ 16.2)

Last time: C oriented curve in \( \mathbb{R}^n \)
\( \vec{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) vector field

\[
\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt
\]

for any parameterization \( \vec{r} : [a, b] \rightarrow \mathbb{R}^n \) of \( C \).

Example: Work = \( \int_C \vec{F} \cdot d\vec{r} \)

[Other examples from E+M, see HW.]

Alternate viewpoint.

Unit tangent vectors

\[
\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}
\]

Example:

\[
\vec{r}(t) = (t, t^2) \quad \vec{r}'(t) = (1, 2t)
\]

\[
|\vec{r}'(t)| = \sqrt{1 + 4t^2}
\]

\[
\vec{T}(t) = \left( \frac{1}{\sqrt{1 + 4t^2}}, \frac{2t}{\sqrt{1 + 4t^2}} \right)
\]
\[ \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt = |\vec{r}'(t)| \vec{\tau}(t) \]

\[ = \int_a^b (\vec{F}(\vec{r}(t)) \cdot \vec{\tau}(t)) |\vec{r}'(t)| \, dt \]

\[ = \int_C \vec{F} \cdot \vec{\tau} \, ds \text{ [Integral of a fn with respect to arc length]} \]

Note that \( \vec{\tau} \) doesn't change if we use a different parameterization, unless we travel the other way along \( C \) and then \( \vec{\tau}(t) \rightarrow -\vec{\tau}(t) \).

Thus if \(-C = C \) oriented the other way have:

\[ \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{\tau} \, ds \]

Integrals of functions don't depend on orientation:

\[ = \int_{-C} \vec{F} \cdot \vec{\tau} \, ds \]

\[ = -\int_{-C} \vec{F} \cdot \vec{U} \, ds = -\int_{-C} \vec{F} \cdot d\vec{r} \text{ [So switching orientation of } C \text{ changes the sign of } \int_C \vec{F} \cdot d\vec{r} \text{]} \]
Alternate notation: C curve in $\mathbb{R}^2$, parameterized by $\vec{r} : [a, b] \to \mathbb{R}^2$, and $\vec{F} : \mathbb{R}^2 \to \mathbb{R}^2$ a vector field

$$\vec{r}(t) = (x(t), y(t)) = x(t) \hat{\imath} + y(t) \hat{j}$$

$$\vec{F}(x, y) = P(x, y) \hat{\imath} + Q(x, y) \hat{j}$$

Then

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) \, dt$$

$$= \int_a^b \left( P(\vec{r}(t)) x'(t) + Q(\vec{r}(t)) y'(t) \right) \, dt$$

$$= \int_a^b P(\vec{r}(t)) x'(t) \, dt + \int_a^b Q(\vec{r}(t)) y'(t) \, dt$$

$$= \int_C P \, dx + \int_C Q \, dy \quad \text{[New notation.]}$$

$$= \int_C P \, dx + Q \, dy$$

*Ex*: Evaluate: \( \int_C y \, dx + dy \)

\[ \vec{r}(t) = (t, t^2) \quad 0 \leq t \leq 1 \]

\[ x'(t) dt + y'(t) dt = \int_0^1 t^2(1 \, dt) + 1(2t) \, dt \]
\[ \int_{0}^{1} t^2 + 2t \, dt = \left. \frac{t^3}{3} + t^2 \right|_{t=0}^{1} = \frac{4}{3}. \]

[Same as at end of last time.]

Similarly in \( \mathbb{R}^3 \): \[ \oint_{C} P \, dx + Q \, dy + R \, dz \]
is just \[ \int_{C} \vec{F} \cdot d\vec{r} \]
where \( \vec{F} = P \vec{i} + Q \vec{j} + R \vec{k} \)

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**Fundamental Theorem of Calculus**: \( f: [a, b] \to \mathbb{R} \)
which is differentiable. Then \[ \int_{a}^{b} f'(t) \, dt = f(b) - f(a). \]

**Fundamental Theorem of Line Integrals**: Suppose \( f: \mathbb{R}^n \to \mathbb{R} \) is differentiable and \( C \) a curve in \( \mathbb{R}^n \)
from \( A \) to \( B \). Then \[ \int_{C} \nabla f \cdot d\vec{r} = f(B) - f(A) \]

**Reason**: Pick \( \vec{r}: [a, b] \to C \)
where \( \vec{r}(a) = A \) and \( \vec{r}(b) = B \).

Then \[ \int_{C} \nabla f \cdot d\vec{r} = \int_{a}^{b} \nabla f(\vec{r}(t)) \cdot \vec{r}'(t) \, dt \]
\[
\int_a^b (D(\vec{r}(t)) f)(\vec{r}(t)) \, dt
\]

rate \( f \) changes as we move along curve at \( t \) time.

\[
= \int_a^b \left( \text{rate of change in } f(t) \right) \, dt = f(\vec{r}(b)) - f(\vec{r}(a))
\]

\[
= f(B) - f(A).
\]

\( \text{Ex: } f(x,y) = x + y \quad \vec{F} = \nabla f = (1,1) \)

\[
\int_C \vec{F} \cdot d\vec{r} = \int_{-\pi/2}^{\pi/2} (1,1) \cdot (-\sin t, \cos t) \, dt
\]

\[
= \int_{-\pi/2}^{\pi/2} -\sin t + \cos t \, dt
\]

\[
= \mid_{t=-\pi/2}^{t=\pi/2} \cos t + \sin t = 1 - (-1) = 2.
\]

Fund. Thm. says as \( \vec{F} = \nabla f \)

\[
\int_C \vec{F} \cdot d\vec{r} = f(B) - f(A) = 1 - (-1) = 2. \checkmark
\]
Consequence: If $\vec{F}$ is conservative ($= \nabla f$) then \( \int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} \) if $C_1$ and $C_2$ have the same end points.

"Independence of path."

Next two lectures will explore conservative vector fields and characterize them by properties of their path integrals.