Lecture 18: Vector fields (§16.1 and 16.2)

Last time:

\[ \int_C f \, ds = \int_a^b f(\hat{r}(t)) |\hat{r}'(t)| \, dt \]

where \( \hat{r} : [a,b] \to \mathbb{R}^2 \) is a parameterization of \( C \)

Note: \( ds = |\hat{r}'(t)| \, dt \) is the arc-length element

Also \( \int_C 1 \, ds = \text{Length}(C) \)

Uses:

1. Average of \( f \) on \( C \): \( \frac{1}{\text{len}(C)} \int_C f \, ds \)
2. Total mass: \( \int_C f \, ds \)
3. Area of the graph of \( f \)

[Everything works for curves in \( \mathbb{R}^3 \) except use 3. Will learn more about diff param. of the same \( C \) in section.]

Vector fields (§16.1 and 16.2)

For \( \mathbb{R}^2 \), a vector field is a function \( \tilde{F} : \mathbb{R}^2 \to \mathbb{R}^2 \)

Ex: \( \tilde{F}(x,y) = -\frac{y}{2} i + \frac{x}{2} j \)

Uses: • Wind speed/direction • Fluid flow • Force magnitude/direction • Electric/magnetic fields

\[ \mathbb{R}^2 \]
Ex: Gravity
Large mass $M$ at $(0,0)$. Force $\vec{F}$ on small mass $m$ depends on position $\vec{r} = (x, y)$ and points in direction $-\vec{r}$.

**Newton's Law:** $|\vec{F}| = \frac{MmG}{|\vec{r}|^2}$

As $\vec{F} = -C\vec{r}$, have $|\vec{F}| = C|\vec{r}| \Rightarrow C = \frac{MmG}{|\vec{r}|^3}$

So $\vec{F} = -\frac{MmG}{|\vec{r}|^3}\vec{r}$

[For several bodies, add vector fields. An electric field generated by a charged particle is similar.]

[Where have we seen vector fields before in this class?]

Ex: Gradients: If $f: \mathbb{R}^2 \to \mathbb{R}$ then $\nabla f: \mathbb{R}^2 \to \mathbb{R}^2$

1. $f(x, y) = x^2 + y^2$
   $\nabla f = (2x, 2y)$

2. $f(x, y) = \frac{MmG}{\sqrt{x^2 + y^2}} = \frac{MmG}{|\vec{r}|}$. Then
\[ \nabla f = MmG \left( -\frac{1}{2} (x^2+y^2)^{-3/2}, 2x, \quad \right) \]
\[ = -MmG \left( \frac{x}{(x^2+y^2)^{3/2}}, \frac{y}{(x^2+y^2)^{3/2}} \right) \]
\[ = -\frac{MmG}{|\vec{r}|^3} \hat{r} = \vec{F} \]

In general, when \( \vec{F} = \nabla f \) we call \( f \) a potential function for \( \vec{F} \) and say \( \vec{F} \) is a conservative vector field. [Think potential energy/conservation of energy.]

Q: Is \( \vec{F} = -\frac{y}{2} \hat{i} + \frac{x}{2} \hat{j} \) conservative?

A: No. Suppose \( \vec{F} = \nabla f \). Since \( \vec{F} \) is tangent to the unit circle, following the circle increases \( f \). But going all the way around, we end up back at \( (1,0) \).
Integrating Vector Fields (§ 16.2)

Work done by gravity: \( W = \mathbf{F} \cdot \mathbf{d} \)

[Assumes constant force.]

How much work does gravity do here?

Motion of ship \( \mathbf{r} : \mathbb{R} \to \mathbb{R}^2 \)

Force of gravity: \( \mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2 \)

Break into segments:

Work done here \( \approx \mathbf{F}(\mathbf{r}(t_i)) \cdot (\mathbf{r}(t_{i+1}) - \mathbf{r}(t_i)) \approx \Delta t \mathbf{F}(\mathbf{r}(t_i)) \cdot \mathbf{r}'(t_i) \approx (\mathbf{F}(\mathbf{r}(t_i)) \cdot \mathbf{r}'(t_i)) \Delta t \)

Sum up and take \( \Delta t \to 0 \) to learn

Total Work \( = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt \)
General Setup: \( C \) curve in \( \mathbb{R}^n \)
\[ \mathbf{F}: \mathbb{R}^n \to \mathbb{R}^n \text{ a vector field} \]
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{\hat{r}}(t)) \cdot \mathbf{\hat{r}}'(t) \, dt \]
for any parameterization \( \mathbf{\hat{r}}: [a, b] \to \mathbb{R}^n \) of \( C \).

[Note: Answer only depends on direction of param.]

Ex: \( C \) \hspace{1cm} \( \mathbf{\hat{r}}(t) = (t, t^2) \) for \( 0 \leq t \leq 1 \)
\( \mathbf{\hat{r}}'(t) = (1, 2t) \)

For \( \mathbf{F} = (y, 1) \) we have
\[ \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{\hat{r}}(t)) \cdot \mathbf{\hat{r}}'(t) \, dt = \int_0^1 (t^2, 1) \cdot (1, 2t) \, dt \]
\[ = \int_0^1 t^2 + 2t \, dt = \left[ \frac{t^3}{3} + t^2 \right]_0^1 = \frac{4}{3} \]

Here's why this is consistent with the work interpretation of \( \int_C \mathbf{F} \cdot d\mathbf{r} \).