Lecture 11: The Chain Rule (§14.5) and Directional Derivatives (§14.6)

Last time: Chain Rule

1. \( f: \mathbb{R}^2 \to \mathbb{R}, \ x, y: \mathbb{R} \to \mathbb{R} \)

For \( h(t) = f(x(t), y(t)) \), we have

\[
    h'(t) = f_x(x(t), y(t))x'(t) + f_y(x(t), y(t))y'(t)
\]

2. \( z = f(x, y) \) with \( x = x(t) \) and \( y = y(t) \). Then

\[
    \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

What if \( x \) and \( y \) themselves are functions of more than one variable?

Ex: \( z = f(x, y) = x^2 + 3y \)

\[
    x(r, \theta) = r \cos \theta \\
    y(r, \theta) = r \sin \theta
\]

So \( z(r, \theta) = f(x(r, \theta), y(r, \theta)) = r^2 \cos^2 \theta + 3r \sin \theta \)

Chain Rule:

\[
    \frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial \theta}
\]

\[
    = 2x \cdot (-r \sin \theta) + 3 \cdot (r \cos \theta)
\]

\[
    = (2r \cos \theta)(-r \sin \theta) + 3r \cos \theta = -2r^2 \cos \theta \sin \theta + 3r \cos \theta
\]
Directional Derivatives (§14.6)

Already have $\partial$-derivatives, measuring rates of change in $x$ and $y$ directions. What about other directions?

Pick a point $(a, b)$ in $\mathbb{R}^2$ and a vector $\vec{v}$.

The derivative of $f$ in direction $\vec{v}$ at $(a, b)$ is

$$D_{\vec{v}} f(a, b) = \frac{\text{rate of change of } f \text{ as we move in direction } \vec{v}}{\text{function of one variable}}$$

$$= \frac{d}{dt} \left. \frac{f(\vec{v}_0 + t\vec{v})}{t} \right|_{t=0}$$

Ex: For $\vec{v} = \vec{c}$ have $D_{\vec{c}} f(a, b) = \frac{\partial f}{\partial x}(a, b)$.

In general, compute using the Chain Rule:

Have $\vec{v}_0 + t\vec{v} = (a + tv_1, b + tv_2)$, so $\vec{v}_0 = (a, b)$, $f(\vec{v}_0 + t\vec{v}) = f(x, y)$ where

$\vec{v} = (v_1, v_2)$

$x = a + tv_1$

$y = b + tv_2$
Now
\[
\mathbf{D} f(a,b) = \frac{df}{dt}(0) = \frac{\partial f}{\partial x}(x(0), y(0)) x'(0)
+ \frac{\partial f}{\partial y}(x(0), y(0)) y'(0)
\]
\[
= \frac{\partial f}{\partial x}(a,b) \mathbf{v}_1 + \frac{\partial f}{\partial y}(a,b) \mathbf{v}_2
\]

Ex: \( f(x,y) = x^2 + y^3 \) \( \mathbf{v} = \frac{1}{\sqrt{2}} \mathbf{i} - \frac{1}{\sqrt{2}} \mathbf{j} \)

(Here, \( \mathbf{v} \) is a unit vector; usually, we take directional derivatives in unit directions since \( \mathbf{D} f(a,b) = 2 \mathbf{D} f(a,b) \).)

\[
\mathbf{D} f(2,1) = \frac{\partial f}{\partial x}(2,1) \cdot \frac{1}{\sqrt{2}} + \frac{\partial f}{\partial y}(2,1) \cdot \left(-\frac{1}{\sqrt{2}}\right)
\]
\[
= 4 \cdot \frac{1}{\sqrt{2}} + 3 \left(-\frac{1}{\sqrt{2}}\right) = \sqrt{2} \approx 0.7071
\]

Gradient: For \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \), define
\[
\nabla f(a,b) = \left( \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right)
\]

Ex: For \( f = x^2 + y^3 \), \( \nabla f(2,1) = (4,3) \)

[Will explain the geometric meaning shortly,]

[but for now notice:
\[
\mathbf{D} f(a,b) = \nabla f(a,b) \cdot \mathbf{v}
\]
Q: In what direction does $f$ increase fastest at $(a, b)$?

[Why might we need to know this?]

Suppose $\hat{v}$ is a unit vector. Then

$$D_{\hat{v}}f(a, b) = \nabla f(a, b) \cdot \hat{v} = |\nabla f(a, b)| \cos \theta$$

To maximize, want $\theta = 0$, that is

$$\hat{v}_{\text{max}} = \frac{\nabla f(a, b)}{|\nabla f(a, b)|}, \quad \text{Note also that} \quad D_{\hat{v}_{\text{max}}}f(a, b) = |\nabla f(a, b)|$$

Summary: $\nabla f(a, b)$ points in the direction that $f$ increases fastest at $(a, b)$. Its length is the rate of said increase.

Ex: $f(x, y) = 1 - x^2 - 4y^2 \quad \nabla f = (-2x, -8y)$

Level sets: $f = 0 : 1 - x^2 - 4y^2 = 0 \iff x^2 + 4y^2 = 1$

$f = -3 : 1 - x^2 - 4y^2 = -3 \iff x^2 + 4y^2 = 4$

$$\iff \frac{x^2}{4} + y^2 = 1.$$

Gradient: $\nabla f(1, 0) = (-2, 0) \quad \nabla f(0, 1) = (0, -8)$
Q: What is $\nabla f(0,0)$?  
A. $\hat{0}$

Morals:

- A min/max can only occur when $\nabla f = \hat{0}$.
- $\nabla f$ is always at right angles to the level sets.