Lecture 10: Chain Rule (§ 14.5)

Last time: \( f: \mathbb{R}^2 \to \mathbb{R} \) is differentiable at \((a, b)\) if

\[
f(a+h, b+k) = f(a, b) + \frac{\partial f}{\partial x}(a, b) \, h + \frac{\partial f}{\partial y}(a, b) \, k + E(h, k)
\]

where

\[
\lim_{(h,k) \to (0,0)} \frac{E(h,k)}{\sqrt{h^2+k^2}} = 0
\]

Thm: If \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \) exist and are continuous near \((a, b)\)

the \( f \) is differentiable at \((a, b)\).

Ex: \( f(x, y) = -x^2 - y^2 \quad f_x = -2x \quad f_y = -2y \)

\( \implies f \) is differentiable everywhere.

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Alt. notation: \( \Delta x = h = x-a \) \[ \text{Read in words} \]

\( \Delta y = k = y-b \)

\( \Delta f = f(x,y) - f(a,b) \approx f_x(a,b) \Delta x + f_y(a,b) \Delta y \)

\( \uparrow \) approximately

You all know:

\[
\frac{d}{dt} \sin(t^2) = \cos(t^2) \cdot (2t)
\]

[Today, will discuss the analog for several variables.]
Setup: For $f, g: \mathbb{R} \to \mathbb{R}$, consider the composition $h = f \circ g$, where $h(t) = f(g(t))$.

Example: $f(t) = \cos(t)$, $g(t) = t^2 \Rightarrow h(t) = \cos(t^2)$

Chain Rule: $h'(t) = f'(g(t))g'(t)$.

1. $f: \mathbb{R}^2 \to \mathbb{R}$

2. Parameterized curve $(x(t), y(t))$

3. Compose them: $h(t) = f(x(t), y(t))$ so $h: \mathbb{R} \to \mathbb{R}$

Example: $f = \text{temperature as a function of position}$.

$h = \text{temperature as a function of time}$.

Q: What is $h'(t)$? [In terms of the derivatives of $x, y$ and the partials of $h$.]

That is, we want to find the linear approximation

$$h(t + \Delta t) = h(t) + h'(t)\Delta t + \frac{E(\Delta t)}{\Delta t}$$ small compared to $\Delta t$. 

Know

\[ X(t + \Delta t) = X(t) + X'(t) \Delta t + E_1(t) \]
\[ y(t + \Delta t) = y(t) + y'(t) \Delta t + E_2(t) \]

and

\[ f(x + \Delta x, y + \Delta y) \approx f(x, y) + f_x(x, y) \Delta x + f_y(x, y) \Delta y \]

So plug in and get

\[ h(t + \Delta t) = f(X(t + \Delta t), y(t + \Delta t)) \]
\[ = f\left( x(t) + x'(t) \Delta t + E_1(t), y(t) + y'(t) \Delta t + E_2(t) \right) \]
\[ \approx f(x(t), y(t)) + f_x(x(t), y(t)) (x'(t) \Delta t + E_1(t)) \]
\[ + f_y(x(t), y(t)) (y'(t) \Delta t + E_2(t)) \]
\[ \approx h(t) + (f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t)) \Delta t \]

where I’ve thrown away all the terms with \( E_1(t) \) and \( E_2(t) \).

**Chain Rule:** \( h(t) = f(x(t), y(t)) \). Then

\[ h'(t) = f_x(x(t), y(t)) x'(t) + f_y(x(t), y(t)) y'(t) \]

[Read in words.]
Ex: \( f(x, y) = (x + y^2)^2 \)

\[
\begin{align*}
X(t) &= \sqrt{2} \cos t \quad y(t) = \sqrt{2} \sin t \\
h(t) &= f(X(t), y(t))
\end{align*}
\]

Find \( h'(\pi/4) \): \( X(\pi/4) = y(\pi/4) = 1 \)

\[
\begin{align*}
X'(t) &= -\sqrt{2} \sin t \quad X'(\pi/4) = -1 \\
y'(t) &= +\sqrt{2} \cos t \quad y'(\pi/4) = 1
\end{align*}
\]

\[
\begin{align*}
f_x &= 2(x + y^2) \quad f_y = 2(x + y^2)(2y) = 4y(x + y^2)
\end{align*}
\]

So:

\[
h'(\pi/4) = f_x(X(\pi/4), y(\pi/4)) \cdot X'(\pi/4) \\
+ f_y(X(\pi/4), y(\pi/4)) \cdot y'(\pi/4)
\]

\[
= f_x(1, 1)(-1) + f_y(1, 1)(1)
\]

\[
= 4(-1) + 8(1) = 4.
\]

Note: \( h(t) = (2 \cos t + 2 \sin^2 t)^2 \)

so you can double-check directly in this case...
Ex: \( f(x, y) = (x + 3y)^2 \)

\[ x(t) = t^2 + 1 \]
\[ y(t) = t \]
\[ h(t) = f(x(t), y(t)) \]

Find \( h'(0) \) using the Chain Rule:

\[ x'(t) = 2t \quad x'(0) = 0 \]
\[ y'(t) = 1 \quad y'(0) = 1 \]

\[ f_x = 2(x + 3y) \quad f_y = 2(x + 3y)^2 \cdot 3 = 6(x + 3y)^2 \]

So:

\[ h'(0) = f_x(x(0), y(0)) \cdot x'(0) + f_y(x(0), y(0)) \cdot y'(0) \]
\[ = f_x(1, 0) \cdot 0 + f_y(1, 1) \cdot 1 \]
\[ = 0 + 6 \cdot 1 = 6. \]

Note: \( h(t) = ((t^2 + 1) + 3t)^2 = (t^2 + 3t + 1)^2 \)

So can double check this directly...
Alternate viewpoint:

$f, g: \mathbb{R} \rightarrow \mathbb{R}$ and $h(t) = f(g(t))$

Set $y = f(x)$ and $x = g(t)$ so that $y$ can be viewed as a function of $t$. Then

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad "\text{Cancel the } dx \text{'s}"$$

Compare

$$\frac{dh}{dt}(t) = \frac{df}{dx}(g(t)) \cdot \frac{dg}{dt}(t)$$

$$= f'(g(t)) \cdot g'(t)$$

Now suppose

$$h(t) = f(x(t), y(t))$$

Chain Rule: $\frac{dh}{dt} = \frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} \quad "\text{Cancel } \partial x \text{ with } dx."$

Sometimes write $z = f(x, y)$ and $x = x(t), y = y(t)$ so that $z$ becomes a function of $t$ and:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$
Also works for \( f: \mathbb{R}^n \to \mathbb{R} \) for \( n > 2 \):

Ex: \( w = f(x, y, z) = x^2 + yz \)
\[ x(t) = t \]
\[ y(t) = t^2 \]
\[ z(t) = 1 - t \]

\[
\frac{dw}{dt} = \frac{\partial w}{\partial x} \frac{dx}{dt} + \frac{\partial w}{\partial y} \frac{dy}{dt} + \frac{\partial w}{\partial z} \frac{dz}{dt}
\]
\[ = 2x \cdot 1 + z \cdot (2t) + y(-1)
\]
\[ = 2t + (1-t)(2t) + t^2(-1)
\]
\[ = 4t - 3t^2
\]

Check: \( w(t) = t^2 + t^2(1-t) = 2t^2 - t^3 \)
\[ w'(t) = 4t - 3t^2. \]