Last time: An eigenvector for a linear operator $T$ on $V$ is a $v \in V$ where $T(v) = \lambda v$ for some scalar $\lambda$. An eigenvector for $A \in \text{M}_{n \times n}(\mathbb{R})$ is a $v \in \mathbb{R}^n$ with $Av = \lambda v$. [Equivalently, it's an eigenvector for $L_A$.]

Today will explain how to find all eigenvectors for $A$. Recall:

Thm: Suppose $A \in \text{M}_{n \times n}(\mathbb{R})$. Then $\lambda \in \mathbb{R}$ is an eigenvalue for $A$ if and only if $\det(A - \lambda I_n) = 0$.

Def: The characteristic polynomial of $A \in \text{M}_{n \times n}(\mathbb{R})$ is $f(t) = \det(A - tI_n)$.

Ex: \[
A = \begin{pmatrix}
0 & 0 & 1 \\
0 & 2 & 0 \\
-2 & 0 & 3
\end{pmatrix}
\quad A - (t \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}) = \begin{pmatrix}
-t & 0 & 1 \\
0 & 2-t & 0 \\
-2 & 0 & 3-t
\end{pmatrix}
\]

$\det(A - t) = (2-t)(-t(3-t) + 2)$

$= (2-t)(t^2 - 3t + 2) = -t^3 + 5t^2 - 8t + 4$
Notes: Suppose $f(t)$ is the characteristic poly of $A \in M_{n \times n}(\mathbb{R})$.

1) For any $\lambda \in \mathbb{R}$, have $f(\lambda) = \det(A - \lambda I_n)$. Thus

Thm: The roots of $f(t)$ are exactly the eigenvalues of $A$.

Ex: $A$ as before, have char poly

$$f(t) = (2-t)(t^2-3t+2)$$
$$= -(t-2)(t-2)(t-1)$$
$$= -(t-1)(t-2)^2$$

So the eigenvalues of $A$ are 1 and 2.

2) $f(t)$ has degree $n$ and leading coefficient $(-1)^n$.

Cor: An $n \times n$ matrix has at most $n$ distinct eigenvalues.
Lemma (Last time) \( \forall \in \mathbb{R}^n \) is an eigenvector for \( A \in M_{n \times n}(\mathbb{R}) \iff \forall \in \mathbb{R}^n \) and \( \forall \neq 0 \).

Ex: Find all eigenvectors of \( A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 3 \end{pmatrix} \)

with eigenvalue 1. Set \( B = A - I_3 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ -2 & 0 & 2 \end{pmatrix} \)

As \( B \xrightarrow{-2R_1 + R_3} \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \), have one free var and

\[ \mathcal{N}(B) = \{ (t, 0, t) \mid t \in \mathbb{R} \} \]

= span \{ (1, 0, 1) \}

So the eigenvectors of \( A \) with eigenvalue 1 are \( (t, 0, t) \) with \( t \neq 0 \).

Check: \( \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \checkmark \)

Def: If \( \lambda \) is an eigenvalue of \( A \in M_{n \times n}(\mathbb{R}) \), define \( E_\lambda = \mathcal{N}(A - \lambda I_n) \), which is called the eigenspace of \( A \) corresponding to \( \lambda \).
Note: \( E_\lambda = \{ \text{all eigenvectors of } A \} \cup \{ 0 \} \)

and \( E_\lambda \) is a subspace of \( \mathbb{R}^n \).

\[ \text{Ex: Compute } E_2 \text{ for } A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ -2 & 0 & 3 \end{pmatrix}, \quad \begin{pmatrix} s \\ t \end{pmatrix} \]

Set \( B = A - 2I_3 = \begin{pmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ -2 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)

\[ E_2 = \mathcal{N}(B) = \{ (t/2, s, t) \mid s, t \in \mathbb{R} \} \]

Q: Is \( A \) diagonalizable? That is, can we find a basis \( \beta \) of \( \mathbb{R}^3 \) so that \([L_A]_\beta \) is a diagonal matrix?

A. Yes is equivalent to having a basis of \( \mathbb{R}^3 \) consisting of eigenvectors.

\[ \begin{align*}
V_1 &= (1, 0, 1) \quad \in E_1 \\
V_2 &= (0, 1, 0) \quad \in E_2 \\
V_3 &= (1, 0, 2) \quad \in E_2
\end{align*} \]
Then $\beta = \{v_1, v_2, v_3\}$ are a basis as if
\[ Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \text{ then } \det(Q) = 1 \neq 0. \]

So
\[
[L_A]_{\beta} = [I_{\mathbb{R}^3}]_{\text{std}}^{-1} [L_A]_{\text{std}} [I_{\mathbb{R}^3}]_{\beta} = Q^{-1} A Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}
\]

and hence $A$ is diagonalizable.

Note: $A \in M_{n \times n}(\mathbb{R})$ is diagonalizable if and only if $A$ is similar to a diagonal matrix.

Today focused on matrices, but can also consider

**Def:** Suppose $T$ is a linear operator on $V$.

If $\lambda$ is an eigenvalue of $T$, then set $E_\lambda = \{v \in V \mid T(v) = \lambda v \}$, which
is again called the eigenspace of $T$ corresponding to $\lambda$. Note $E_\lambda$ is a subspace of $V$ and consists of all eigenvectors with this eigenvalue, plus 0.

Exercise: $E_\lambda(A) = E_\lambda(L_A)$. 

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