Last time: \( A \in M_{n \times n}(\mathbb{R}) \):

1. \( A \xrightarrow{R_r \leftrightarrow R_s} B \) \( \Rightarrow \) \( \det(B) = -\det(A) \)
2. \( A \xrightarrow{cR_r} B \) \( \Rightarrow \) \( \det(B) = c \det(A) \)
3. \( A \xrightarrow{cR_s + R_r} B \) \( \Rightarrow \) \( \det(B) = \det(A) \)

Today: \( \det(AB) = \det(A) \det(B) \)

\[ \text{Strategy: Relate row ops to matrix multiplication, but first here's one more easy consequence of what we learned last time.} \]

Recall \( \text{rank}(A) = \dim(\text{ColSp}(A)) = \dim(\text{RowSp}(A)) \).

Thm: For \( A \in M_{n \times n}(\mathbb{R}) \), if \( \text{rank}(A) < n \) then \( \det(A) = 0 \).

Proof: As \( \text{rank}(A) < n \), some row is a linear combination of the others, say

\[ a_r = c_1 a_1 + \cdots + c_{r-1} a_{r-1} + c_{r+1} a_{r+1} + \cdots + c_n a_n \]

where \( a_i \) is row \( i \) of \( A \). If we do row ops
- C_i R_i + R_r for i = 1, ..., n and i ≠ r, then we get a matrix B whose r-th row is 0.

Hence by last time \( \text{det}(B) = 0 \). By 3, we have \( \text{det}(A) = \text{det}(B) \) so \( \text{det}(A) = 0 \) as req'd.

**Def:** An non-elementary matrix is the result of doing a single row operation to \( I_n \).

**Ex:**

1. \( I_3 \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

2. \( I_4 \xrightarrow{R_1 \leftrightarrow R_4} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \)

3. \( I_4 \xrightarrow{5R_2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \)

4. \( I_4 \xrightarrow{5R_2 + R_1} \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

5. \( I_4 \xrightarrow{-3R_1 + R_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix} \)

Note that these are the identity matrix except in at most 1 place (2 and 3) or 4 places (1).
Thm: Suppose $E$ is the elementary matrix where $I_n \xrightarrow{R} E$. If $A \in M_{n \times n}(\mathbb{R})$ then $A \xrightarrow{R} EA$.

Proof of Thm: HW #7.

Thm: Every elementary matrix is invertible.

Proof: Suppose $I_n \xrightarrow{R} E$. Let $R'$ be the row operation that reverses $R$, that is $A \xrightarrow{R} B \xrightarrow{R'} A$ for all $A \in M_{n \times n}(\mathbb{R})$. [Query: Why does this exist?]

Let $E'$ be the elementary matrix associated with $R'$. By previous theorem, have

\[
E'E = \text{result of doing } R' \text{ to } E = I_n
\]
\[
EE' = \text{result of doing } R \text{ to } E' = I_n
\]
So $E$ is invertible with inverse $E'$.

**Thm**: $A \in M_{n \times n}(\mathbb{R})$ is invertible if and only if it is the product of elementary matrices.

**Proof**: $(\Leftarrow)$ If $A = E_1 E_2 \cdots E_k$ with $E_k$ elementary, then each $E_k$ is invertible and so

$$A^{-1} = E_k^{-1} E_{k-1}^{-1} \cdots E^{-1}$$

by HW.

$(\Rightarrow)$ If $A$ is invertible, then

$$B = (A : I_n) \xrightarrow{\text{row ops}} (I_n : A^{-1}) = C$$

As each row operation can be implemented by a product by an elementary matrix, we have $E_k$ where

$$E_k \cdots E_2 E_1 B = C$$

which implies

$$E_k E_{k-1} \cdots E_2 E_1 A = I_n$$
and so

\[ A = E^{-1}_e \cdots E^{-1}_2 E^{-1}_1 \]

As the $E^{-1}_k$ are also elementary, we're done. \(\square\)

**Theorem:** \( \det(AB) = \det(A) \det(B) \).

**Proof:** If \( \text{rank}(AB) < n \) then \( \det(AB) = 0 \).

Moreover, one of \( A, B \) must have rank \( < n \) and so one of \( \det(A) \), \( \det(B) \) is \( 0 \). So

\[ \det(AB) = \det(A) \det(B) \] in this case.

So we have reduced to the case where \( A, B, \) and \( AB \) all have rank \( n \). In particular

\[ A = E_1 \cdots E_e \quad B = E_{l+1} \cdots E_m \]

where \( E_k \) are elementary. The result now follows from
Claim: Suppose $C = E'_1 \ldots E'_p$ where $E'_k$ are elementary. Then

$$\det(C) = (-1)^{\text{(# of type 1 $E'_k$)}} \cdot (\text{product of } C_k \text{ in all type 2 } E'_k)$$

Proof of Claim: $C$ is obtained from $I_n$, which has $\det 1$, by the row ops $R'_p, R'_{p-1}, \ldots, R'_1$.

By last time, only the type 1 and 2 ops change the det and moreover do so in a way that proves the claim.

Cor: For $A \in M_{n \times n}(\mathbb{R})$, have $\det(A) \neq 0$ if and only if $A$ is invertible.

Proof: If $A$ is invertible, then $1 = \det(I_n) = \det(A \cdot A^{-1}) = \det(A) \cdot \det(A^{-1}) \Rightarrow \det(A) \neq 0$.

If instead $A$ is not invertible, then $\det(A) = 0$ by first result of today.