Lecture 4: Linear combinations and systems of linear equations

[§1.4 of FIS] and

[§5.5LE of B]

Last time: A subspace \( W \) of a vector space \( V \) is something where:

@ \( 0 \) is in \( W \) ② \( W \) is closed under addition.
③ \( W \) is closed under scalar multi.

A linear combination of vectors \( u_1, u_2, \ldots, u_n \) in \( V \) is any vector of the form \( a_1 u_1 + a_2 u_2 + \ldots + a_n u_n \) where the \( a_i \) are in \( \mathbb{R} \).

Ex: \( V = \mathbb{R}^3 \)

\[ u_1 = (0, -1, 0) \]
\[ u_2 = (1, 0, -1) \]

Some linear combinations.
The span of vectors \( u_1, \ldots, u_n \) in \( V \) is the set of all linear combinations of \( u_1, \ldots, u_n \).

Ex: \( \text{span}(u_1, u_2) = \{ a_1 u_1 + a_2 u_2 \mid a_1, a_2 \in \mathbb{R} \} = \{(a_1, -a_2, -a_1)\} \)

= plane \( W \) in \( \mathbb{R}^3 \) given by \( \{ x + z = 0 \} \).

Thm: The span of any \( u_1, \ldots, u_n \) in \( V \) is a subspace of \( V \).

Pf: (a) \( O = 0 u_1 + \cdots + 0 u_n \) is in \( V \)

(b) \( (a_1 u_1 + \cdots + a_n u_n) + (b_1 u_1 + \cdots + b_n u_n) \)

\[ = (a_1 + b_1) u_1 + \cdots + (a_n + b_n) u_n \]

\[ \uparrow \text{[Query: Why can I do this?] \]}

(c) For \( c \) in \( \mathbb{R} \) have

\[ c \cdot (a_1 u_1 + \cdots + a_n u_n) = (ca_1) u_1 + \cdots + (ca_n) u_n . \]
Ex: $V = \mathbb{R}^3$ 
$u_1 = (1, 1, -1)$
$u_2 = (-1, 1, 2)$

\[ W = \text{span}(u_1, u_2) \]

Q1: Find eqns for $W$ of the form $c_1 x + c_2 y + c_3 z = 0$

Know \[ c_1 \cdot 1 + c_2 \cdot 1 + c_3 (-1) = 0 \] since $u_1, u_2$
\[ c_1 (-1) + c_2 \cdot 1 + c_3 \cdot (2) = 0 \] are in $W$.

This gives the following system of eqns

\[ c_1 + c_2 - c_3 = 0 \]
\[ -c_1 + c_2 + 2c_3 = 0 \]

Add top
\[ 2c_2 + c_3 = 0 \]

Add bottom
\[ c_1 + 3c_2 = 0 \]

If we take $c_2 = -1$, get \[ c_1 = 3, \ c_3 = 2. \]

So equation for plane is \[ 3x - y + 2z = 0. \]

[Double check!]
Q2: \( \mathbf{v} = (5, 1, -7) \) is in \( W \) as it satisfies the eqn. Thus it must be a linear combination of \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \), i.e.

\[
\mathbf{v} = a_1 \mathbf{u}_1 + a_2 \mathbf{u}_2 = (a_1 - a_2, a_1 + a_2, -a_1 + 2a_2)
\]

leading to a system of three equations:

\[
\begin{align*}
a_1 - a_2 &= 5 \\
a_1 + a_2 &= 1 \quad \quad \quad \quad \quad \quad \quad \text{add} \implies 3a_2 &= -6 \\
-a_1 + 2a_2 &= -7 \\
\end{align*}
\]

\( \implies a_2 = -2 \)

and so \( a_1 = 1 - a_2 = 3 \). Check:

\[
3 \cdot \mathbf{u}_1 - 2 \cdot \mathbf{u}_2 = (3, 3, -3) + (2, -2, -4) = \mathbf{v}
\]

Q2: \( \mathbf{w} = (6, 1, -7) \) is not in \( W \). If we try to write \( \mathbf{w} \) as a combination of \( \mathbf{u}_1 \) and \( \mathbf{u}_2 \), we are led to

\[
\begin{align*}
a_1 - a_2 &= 6 \\
a_1 + a_2 &= 1 \\
-a_1 + 2a_2 &= -7
\end{align*}
\]

which still leads to \( a_2 = -2 \) but \( a_1 = 8 \) by
the 1st eqn but \( a_1 = 3 \) by the second one. So there are no solutions to this system, which makes sense geometrically.

System of Linear Equations:

Variables: \( x_1, x_2, \ldots, x_m \)

Equations: \( a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m = b_1 \)
\( a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m = b_2 \)
\[ \vdots \]
\( a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_m = b_n \)

where \( a_{ij} \) and \( b_i \) are real numbers for \( 1 \leq i \leq n \)
\( 1 \leq j \leq m \).

[So no \( x_i^2 \), much less a trig fn.]

Basic tool to solve:

1. Add two equations together, or add a multiple of one eqn to another:
Ex: For

\[
\begin{align*}
2x_1 + x_2 - 3x_3 &= 3 \\
3x_1 - 2x_2 + 6x_3 &= 1
\end{align*}
\]

might take \((\text{Eqn} \ 1) + (\text{Eqn} \ 2) \Rightarrow 7x_1 = 7\)

\[
4x_1 + 2x_2 - 6x_3 = 6
\]

Thus \(x_1 = 1\).

Next time: Develop systematic method for solving such equations.

\[
\begin{bmatrix}
2 & 1 & -3 \\
3 & -2 & 6
\end{bmatrix}
\]

This will be our basic computational tool for the first half of the semester...

Thm: A linear system has either exactly one solution, no solutions, or infinitely many solutions.

Note that we saw this in the 3 examples today.