Lecture 3: Subspaces (§ 1.3 of [FIS])

Previously on Math 416...

A vector space over $\mathbb{R}$ is a set $V$ with two operations (vector addition and scalar mult.) satisfying:

1-2) vector addition is commutative and associative.
3) There is a zero vector.
4) Additive inverses exist.
5) $1v = v$
6) Scalar mult. is assoc.
7-8) Distributive properties.

Ex: $\mathbb{R}^n$, $\text{Mat}_{m\times n}(\mathbb{R})$, spaces of functions...

Back to $\mathbb{R}^3$: Other basic objects: lines and planes.

Today: Analog of such in a general vector space.
Suppose $W$ is a plane in $\mathbb{R}^3$ containing $0$, and $w_1, w_2$ are vectors in $W$. Then $w_1 + w_2$ is also in $W$. So is $aw_1$ for any $a$ in $\mathbb{R}$.

**Note**: Important that $W$ contains $0$ here, as otherwise these props need not hold.

**Def**: Suppose $V$ is a vector space over $\mathbb{R}$. A subset $W$ of $V$ is a subspace if

1. $0$ is in $W$
2. For all $w_1, w_2$ in $W$, the sum $w_1 + w_2$ is also in $W$.
3. For all $c$ in $\mathbb{R}$ and $w$ in $W$, $cw$ is also in $W$.

[Can replace 1 with requirement that $W$ is nonempty.]
Ex: Some subspaces of $\mathbb{R}^3$:

1. $\mathbb{R}^3$
2. $\{0\}$
3. $\{(x,0,0) \text{ for } x \text{ in } \mathbb{R}\}$
4. $\{(x,-x,2x)\}$
5. $\{(x,y,0)\}$
6. $\{x+y+z = 0\} = \{s(1,0,-1)+t(1,-1,0) \text{ for } s,t \text{ in } \mathbb{R}\}$

Ex: In any vector space $V$, the subsets $\{0\}$ and $V$ are subspaces.

Thm: Suppose $W$ is a subspace of a vector space $V$. Then $W$ is itself a vector space under the two operations inherited from $V$.

Proof: First by requirements 2 and 6 we do have two ops taking values in $W$.

Of the 8 conditions, 1-2 and 5-8 are immediate from the fact that $V$ itself is a vector space. Moreover, 3 follows from subspace cond. 2.
Finally, for 4 given \( w \) in \( W \) we know there is a \( v \) in \( V \) such that \( v + w = 0 \).

**Issue:** Does \( v \) have to be in \( W \)?

Yes, since we can take \( w = -(v) \) which is in \( W \) by 5. Check: \( v + (-1)v = 1v + (-1)v = (1-1)v = 0v = 0 \) \( \Rightarrow \) Thm of last time.

So \( W \) with these ops satisfies 8 and so is a vector space.

**Non-Ex:** \( W = \{(w_1, w_2) \text{ with } w_i \geq 0\} \) in \( \mathbb{R}^2 \) is not a subspace.

Satisfies 4 and 6 but not 5.

In proof of them, everything works except 4.

[Discuss difference with book's treatment of subspaces.]
Ex: \( A \in \text{Mat}_{n \times n}(\mathbb{R}) \)

\[
A = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1n} \\
A_{21} & A_{22} & \cdots & A_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n1} & A_{n2} & \cdots & A_{nn}
\end{pmatrix} = (A_{ij})
\]

Transpose: \( A^t \) where \( A^t_{ij} = A_{ji} \)

\[
\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}^t = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 6 \\ 3 & 6 & 9 \end{pmatrix}
\]

[Also works for non-square matrices.]

A matrix \( A \) in \( \text{Mat}_{n \times n}(\mathbb{R}) \) is symmetric if

\( A = A^t \).

Ex: \( \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \) but neither of the two examples above.

Thm: The subset of symmetric matrices in \( \text{Mat}_{n \times n}(\mathbb{R}) \) is a subspace.
Proof: The 0 in Mat\(_{nxn}\)(R) is \( \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \) which is symmetric so 0 holds.

For 1 and 2, first show that for all \( A, B \) in Mat\(_{nxn}\)(R) and \( a, b \) in R one has

\[
(aA + bB)^t = a(A^t) + b(B^t).
\]

Now if \( A, B \) are sym, then

\[
(A + B)^t = A^t + B^t = A + B
\]

and so \( A + B \) is also sym, proving 6.

The argument for 3 is similar. \( \square \)

Next time: Linear combinations and linear equations