Lecture 2: Vector spaces

Ex: Vectors in \( \mathbb{R}^2 \), \( \mathbb{R}^3 \), or indeed \( \mathbb{R}^n \).

Def: A vector space over \( \mathbb{R} \) is a set \( V \) with two operations

Addition: Assigns to each pair \( v, w \) in \( V \) a unique \( v + w \) in \( V \).

Scalar mult: Assigns to each \( a \) in \( \mathbb{R} \) and \( v \) in \( V \) a unique \( av \) in \( V \).

where the following rules hold.

1) For all \( u, v \) in \( V \), \( u + v = v + u \).
2) For all \( u, v, w \) in \( V \), \( (u + v) + w = u + (v + w) \).
3) There is an elt of \( V \), called "0", so that for all \( v \) in \( V \), \( v + 0 = v \).
4) For all \( v \) in \( V \) there exist \( w \) in \( V \) with \( v + w = 0 \).
5) For all $v$ in $V$, $1v = v$.
6) For all $a, b$ in $\mathbb{R}$ and $v$ in $V$, $(ab)v = a(bv)$
7) For all $a$ in $\mathbb{R}$ and $u, v$ in $V$:
   
   \[ a(u + v) = au + bv \]
8) For all $a, b$ in $\mathbb{R}$ and $v$ in $V$, $(a + b)v = av + bv$

**Example:** $\mathbb{R}^n$ with coordinate-wise addition and scalar mult.

[Check one rule, chosen by the class.]

**Example:**

\[
\text{Mat}_{m \times n} = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \mid \text{where } a_{ij} \text{ are in } \mathbb{R} \right\}
\]

where addition and scalar mult are again componentwise.

\[
\begin{pmatrix} 1 & 0 & 3 \\ 0 & 5 & 1 \end{pmatrix} + 2 \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 7 \\ 0 & 3 & 7 \end{pmatrix}
\]

\[
\begin{pmatrix} 2 & 2 & 4 \\ 0 & -2 & 6 \end{pmatrix}
\]
Example: \( f = \{ \text{Continuous functions from } [-1, 1] \text{ to } \mathbb{R} \} \)

- \( f + g \) is the function \((f + g)(x) = f(x) + g(x)\).
- \( af \) is the function \((af)(x) = af(x)\).

Some aspects of vectors in 2 and 3d are not part of this definition (no dot product, for ex.), however, many familiar properties do follow from these rules. For example,

Definition: \( \mathbf{0} \cdot \mathbf{v} = \mathbf{0} \)
\[ \mathbf{0} \in \mathbb{R} \quad \mathbf{v} \in V \]

Thm: If \( u, v, w \) are in a vector space \( V \)

and \( u + w = v + w \), then \( u = v \).

Proof: By (4), there is a \( z \) in \( V \) with

\( w + z = 0 \). So

\[ u = u + 0 = u + (w + z) = (u + w) + z \quad \text{(3)} \]

\[ = (v + w) + z = V + (w + z) = V + 0 = V \quad \text{(3)} \]

Hypothesis
**Thm:** If \( v \) is in a vector space \( V \), then \( 0v = 0 \) in \( V \).

**Proof:** We have

\[
0v + 0v = (0 + 0)v = 0v = 0v + 0
\]

\[\text{(③)}\]

\[\text{(①)}\]

By the previous theorem, this gives

\( 0v = 0 \).

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**Related facts (see text and HW)**

a) The 0 vector is unique.

b) The vector \( w \) with \( v + w = 0 \) is unique; we'll call it "\(-v\)". Note

\[-v = (-1)v \text{ as}\]

\[v + (-1)v = 0 \text{ by above.}\]
Sometimes, we will want to use scalars other than \( \mathbb{R} \), for example the complex numbers \( \mathbb{C} = \{a + bi\} \) where \( a, b \) are in \( \mathbb{R} \) and \( i^2 = -1 \). More generally, we can define a vector space over an arbitrary field \( \mathbb{F} \), which is a set with four operations \( (+, \times, -, ÷) \) satisfying a bunch of axioms. For the first part of this course we will just focus on \( \mathbb{R} \), but the text uses the language of fields.

See Appendix C of [FJS] for more of fields.