Lecture 7: Solution spaces to linear systems

Previously...

Thm: If $M$ and $N$ are row equivalent matrices then the linear systems $L(M)$ and $L(N)$ have the same set of solutions.

Thm: Any matrix is row equivalent to one in reduced row echelon form (RREF),
where:
1. rows of zeros at bottom
2. other rows have leading 1s.
3. a column containing a leading 1 is otherwise 0.
4. leading 1s move down and to the right.

Upshot: To understand solutions to linear systems, it is enough to handle those whose augmented matrix is in RREF.

Ex: \[
\begin{pmatrix}
1 & 2 & 0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3 & 0 & 3 \\
0 & 0 & 0 & 0 & 0 & 4
\end{pmatrix}
\leftrightarrow
\begin{align*}
x_1 + 2x_2 + x_4 &= 2 \\
x_3 + 3x_4 &= 3 \\
x_5 &= 4
\end{align*}
\]
Rewrite as \[ \begin{align*}
X_1 &= 2 - 2X_2 - X_4 \\
X_3 &= 3 - 3X_4 \\
X_5 &= 4
\end{align*} \]

\[ \text{Key: No } X_1, X_3 \text{ or } X_5 \text{ on this side because of rule 3 of RREF.} \]

That is, solve for the variables corresponding to the leading 1s. Will view \( X_2 \) and \( X_4 \) as "free variables".

Thus the solution set to these eqns is: "where" \( X_1 \) "in" \[ \{(2 - 2s - t, s, 3 - 3t, t, 4) \mid s, t \in \mathbb{R}\} \]

Ex: \[
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 3
\end{pmatrix}
\]

\[ \iff \begin{align*}
X_1 &= 2 \\
X_2 &= 3
\end{align*} \]

Sol'n set \( \{ (2, 3) \} \)

Ex: \[
\begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 3/2 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

\[ \iff \begin{align*}
X_1 - X_3 &= 0 \\
X_2 + 3/2X_2 &= 0 \\
0 &= 1
\end{align*} \]

No solutions: Solution set is \( \emptyset \).
A linear system is consistent when it has at least one solution; otherwise it is inconsistent.

A pivot column of a matrix in RREF is one containing a leading 1.

**Thm:** If $M$ is in RREF, then $\mathbb{L}(M)$ is inconsistent if and only if the rightmost column is a pivot column.

**Pf:** If the rightmost column is pivot, we have \[ \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \] and so the lin sys has the eqn $0 = 1$ which has no solutions.

Suppose instead the rightmost column is not pivot. Let $d_1, d_2, \ldots, d_k$ be the indices of the pivot columns, and $b_1, \ldots, b_m$ be the entries of the rightmost column.
Ex: \[
\begin{pmatrix}
0 & 1 & 5 & 0 & 0 & 3 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
\begin{align*}
d_1 &= 2 & b_1 &= 3 \\
d_2 &= 4 & b_2 &= 0 \\
d_3 &= 5 & b_3 &= 2 \\
& & b_4 &= 0 \\
\end{align*}

Claim: \( X d_i = b_i, \ X d_2 = b_2, \ldots, X d_k = b_k \)
and all other \( X j = 0 \) is a solution to \( LS(M) \).

Reason: Each eqn has the form
\[
X d_i + \text{(terms not involving any } X d_k) = b_i.
\]

So when the rightmost column is not pivot \( LS(M) \) has at least one solution,

as desired.

Thm: Any linear system has either no sol'ns, exactly one solution, or infinitely many sol'ns.
Contrast: $x^2 + y^2 = 2$ has exactly two solutions, namely $(1, -1)$ and $(-1, 1)$.

Proof Idea: Can assume that our $m \times (n+1)$ matrix $M$ is in RREF with no zero rows. If column $n+1$ is pivot, then the system has no solutions and we're done. As every row has a leading 1 and the last col is non-pivot, we have $m \leq n$.

Two cases:

$m = n$: In this case, $M = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & b_1 \\
0 & 1 & 0 & \cdots & 0 & b_2 \\
0 & 0 & 1 & \cdots & 0 & b_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & b_m
\end{pmatrix}$

which corresponds to $x_1 = b_1$

$x_n = b_n$ \iff \text{unique solution}.

$m < n$: In this case, there is at least one non-pivot column. If we assign any real values to the non-pivot variables,
we can always solve for the pivot variables as we did before. Thus we get infinitely many solutions in this case.

Note: In the last case, we need $n-m$ "parameters" to write down all possible solutions to the linear system.

Next time: Begin making precise this notion of "number of parameters" in the context of subspaces of a vector space.