Recall: \( V, W \) vector spaces. A function \( T: V \rightarrow W \) is a linear transformation (or just linear) if for all \( v_1, v_2 \in V \) and \( a \in \mathbb{R} \) we have

\begin{enumerate}
  \item \( T(v_1 + v_2) = T(v_1) + T(v_2) \)
  \item \( T(a v_1) = a T(v_1) \)
\end{enumerate}

**Ex:** \( T: \mathbb{R}^2 \rightarrow \mathbb{R}^2 \)

\[
T(x, y) = (-y, x)
\]

notation by \( T/2 \) counterclockwise.

\[
S: \mathbb{R}^2 \rightarrow \mathbb{R}^2
\]

\[
(x, y) \rightarrow (x+y, y)
\]

**Ex:** \( T: \mathcal{C}([0,1]) \rightarrow \mathbb{R} \)

\[
\{ f \in \mathcal{F}([0,1], \mathbb{R}) \mid f \text{ is continuous} \}
\]

\[
T(f) = \int_0^1 f(x) \, dx.
\]

\[
T(x) = \frac{1}{2}, \quad T(x^2) = \frac{1}{3}
\]
That this $T$ is linear is just the basic props of definite integrals:

$$\int_0^1 c f(x) + g(x) \, dx = c \int_0^1 f(x) \, dx + \int_0^1 g(x) \, dx.$$ 

[Heading toward the Dim Thm...]

**Thm:** Suppose $T: V \to W$ is linear. If $\beta = \{v_1, v_2, \ldots, v_n\}$ is a basis for $V$ then $T$ is determined by its values on $\beta$. Moreover

$$\text{range of } T = \{T(v) \mid v \in V\} = \text{Span} \left(\{T(v_1), \ldots, T(v_n)\}\right)$$

**Proof:** Suppose we know $T(v_1), \ldots, T(v_n)$. Given $v \in V$ there are unique scalars such that

$$v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n$$

Repeatedly using props a) and b) for linear transformations, we get

$$T(v) = a_1 T(v_1) + a_2 T(v_2) + \ldots + a_n T(v_n).$$
So the $T(v_i)$ determine $T$. Also we've learned that $R(T) \subseteq \text{span}(\{T(v_i)\})$.

As $T(v_i) \in R(T)$ and $R(T)$ is a subspace, we have $R(T) = \text{span}(\{T(v_i)\})$. So $R(T) = \text{span}(\{T(v_i)\})$ as claimed.

Recall $N(T:V \rightarrow W) = \{v \in V \mid T(v) = 0\}$ is also a subspace.

**Dimension Thm:** Suppose $T:V \rightarrow W$ is linear.

If $V$ is finite dim'l, then

$$\dim(N(T)) + \dim(R(T)) = \dim V$$

nullity of $T$ rank of $T$

[Last time, gave some examples. Here are more.]

**Ex:** $T: \mathbb{R}^{a+b} \rightarrow \mathbb{R}^{a+c}$ has $c$ zeros

$$(x_1, \ldots, x_n) \rightarrow (x_1, \ldots, x_a, 0, \ldots, 0)$$

So $R(T) = \{(x_1, \ldots, x_a, 0, \ldots, 0)\}$
\[ N(T) = \{ (0, \ldots, 0, x_{a+1}, \ldots, x_{a+b}) \mid x_{a+1}, \ldots, x_{a+b} \in \mathbb{R} \} \]

So

\[
\text{nullity} + \text{rank} = b + a = \dim \mathbb{R}^{a+b}.
\]

**Ex:** \( \mathbb{R}^3 \rightarrow \mathbb{R}^2 \)

\[
(x, y, z) \rightarrow (x, y)
\]

**Ex:** \( \mathbb{R}^3 \rightarrow \mathbb{R} \)

\[
(x, y, z) \rightarrow x
\]

**Pf:** [Idea: In the right coordinates (= bases)]

Any linear transformation looks like these exs.

Let \( \beta' \) be a basis for \( N(T) \). By cor of the Repl. Thm, can enlarge this to a basis \( \beta \) of \( V \), say \( \beta = \{ v_1, \ldots, v_a, \overline{x_{a+1}, \ldots, x_{a+b}} \} \)

It's enough to show that \( \beta' \)

\[
\gamma = \{ T(v_1), \ldots, T(v_a) \}
\]

is linearly independent as then by last
Theorem: It is a basis of \( R(T) \) and so

\[
\text{nullity} + \text{rank} = b + \#\gamma = b + a = \#\beta = \dim V.
\]

Suppose have \( \gamma \) with \( c_1 T(v_1) + \cdots + c_a T(v_a) = 0 \)

By linearity of \( T \), have \( \gamma \) is equal to

\[
T(c_1 v_1 + \cdots + c_a v_a).
\]

So \( w = c_1 v_1 + \cdots + c_a v_a \in N(T) \), and so

it is a linear comb. of \( \beta \) with

\[
w = c_{a+1} v_{a+1} + \cdots + c_{a+b} v_{a+b}.
\]

By linear indep of \( \beta \), we must have all \( c_i = 0 \). So \( \gamma \) is linearly independent as needed to prove the theorem.