Lecture 28:

Last time: \( \lambda \) an eigenvalue of \( A \in M_{n \times n} \).

Algebraic Mult: \# of times \((t - \lambda)\) divides char poly of \( A \).

Geometric Mult: \( \dim (E_\lambda) \).

Thm: A matrix \( A \in M_{n \times n}(\mathbb{R}) \) is diagonalizable if and only if:

1) The char poly of \( A \) splits completely over \( \mathbb{R} \).
2) For all eigenvalues of \( A \),

\[(\text{alg mult}) = (\text{geom mult}) . \]

Lemma: Suppose \( v_1, \ldots, v_k \in \mathbb{R}^n \) are eigenvectors of \( A \) corresponding to distinct eigenvalues \( \lambda_1, \ldots, \lambda_k \).

Then \( \{v_1, \ldots, v_k\} \) is linearly independent.

Moral: Can't create an eigenvector with eigenvalue \( \lambda \) from eigenvectors with other eigenvalues.
Ex: $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

Proof of Lemma: Induction on $k$.

**Base case:** As $v_1$ is an eigenvector, $v_1 \neq 0$ and so $\{v_1\}$ is linearly independent.

**Inductive Step:** Assume $\{v_1, \ldots, v_{k-1}\}$ is linearly independent. Suppose there are scalars with

$$a_1 v_1 + a_2 v_2 + \cdots + a_k v_k = 0 \quad (1)$$

Multiplying both sides by $A$ gives

$$a_1 \lambda_1 v_1 + \cdots + a_k \lambda_k v_k = A 0 = 0 \quad (2)$$

Considering $-\lambda_k (1) + (2)$ gives

$$a_1 (\lambda_1 - \lambda_k) v_1 + \cdots + a_{k-1} (\lambda_{k-1} - \lambda_k) v_{k-1} + a_k v_k = 0$$

So for $i < k$ have $a_i (\lambda_i - \lambda_k) = 0$; as $\lambda_i \neq \lambda_k$ this forces $a_i = 0$ for $i < k$.

Thus $(1)$ gives $a_k v_k = 0$ which implies $a_k = 0$.
So all $a_i = 0$ and $\{v_1, \ldots, v_k\}$ is linearly independent, completing the induction. \[\square\]

**Proof of Thm**: $(\Rightarrow)$ By last time, know char poly $A = (\lambda_i - t)^{m_i} \cdots (\lambda_k - t)^{m_k}$ where the $m_i$ are the distinct eigenvalues of $A$ and $\sum m_i = n$. Set $d_i = \dim E_{\lambda_i}$.

Must show each $d_i = m_i$ and already know that $d_i \leq m_i$. Let $\beta$ be a basis of $\mathbb{R}^n$ consisting of eigenvectors for $A$. Set $C_i = \# \{ v \in \beta \mid v \in E_{\lambda_i} \}$. As any subset of $\beta$ is linearly indep, must have $C_i \leq d_i$.

Now

\[ n = \sum C_i \leq \sum d_i \leq \sum m_i = n \]
which forces \( d_i = m_i \) for all \( i \) as req'd.

\[(\Leftarrow)\] Let \( \lambda_i, d_i, m_i \) be as above. As the char poly splits completely, have \( \Sigma m_i = n \),
and by assumption \( m_i = d_i \) for each \( i \).

Let \( \beta_i \) be a basis for \( E_{\lambda_i} \).

Claim: \( \beta = \beta_1 \cup \ldots \cup \beta_k \) is a basis for \( \mathbb{R}^n \).

If so, then \( A \) is diagonalizable as desired.

Now \( E_{\lambda_i} \cap E_{\lambda_j} = \emptyset \) if \( i \neq j \), so

\[
\# \beta = \sum_{i=1}^{k} \# \beta_i = \sum_{i=1}^{k} m_i = n
\]
and thus it suffices to show that \( \beta \) is linearly independent.

Suppose

\[
\beta_i = \{ v_1^i, v_2^i, \ldots, v_{m_i}^i \}
\]
and there are scalars $a^i_j$ where

$$\sum_{i=1}^{k} \left( \sum_{j=1}^{m_i} a^i_j v^i_j \right) = 0$$

$w_i$

Each $w_i$ is either 0 or an eigenvector corr. to $\lambda_i$. By lemma, can't have a linear dependence among eigenvectors with different eigenvalues, so must have all $w_i = 0$. As each $\beta_i$ is linearly independent, must have $a^i_1, \ldots, a^i_{m_i}$ all 0.

So all $a^i_j = 0$ and so $\beta$ is linearly independent. This proves the claim and thus the theorem. Q.E.D.