Convention: Today, $V$ will always be a finite-dimensional inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

**Story so far:** $T$ linear operator on $V$.

- **Normal:** $T \circ T^* = T^* \circ T$.
- **Fact:** When $\mathbb{F} = \mathbb{C}$, any normal $T$ is diagonalizable via an orthonormal basis.

- **Self-adjoint:** $T = T^* \Rightarrow$ diagonalizable via an orthonormal basis even if $\mathbb{F} = \mathbb{R}$.

**Today:** Linear op $T$ on $V$ where

$$\langle T(x), T(y) \rangle = \langle x, y \rangle \text{ for all } x, y \in V.$$  

[Reflect on theme of preserving structure...]

Such $T$ are called **orthogonal** when $\mathbb{F} = \mathbb{R}$

**unitary** when $\mathbb{F} = \mathbb{C}$

I'll use the generic term **isometry** for either field.

**Ex:** $(\mathbb{R}^2, \text{dot})$: rotations, reflections.

$(\mathbb{R}^n, \text{dot})$: rigid motions fixing 0.

**Non Ex:** Anything with $N(T) \neq \emptyset$.  

**Thm:** For a linear op $T$ on $V$, the following are equivalent:

a) $T$ is an isometry.

b) $T^* T = T^* T = I_V$ (⇒ $T$ is normal.)

c) For every orthonormal basis $\beta$ of $V$, the image $T(\beta)$ is also an orthonormal basis.

d) For some orthonormal basis $\beta$, $T(\beta)$ is orthonormal.

e) $\|T(x)\| = \|x\|$ for all $x \in V$.

**Lemma:** Suppose $S$ is a self-adjoint operator on $V$. If $\langle S(x), x \rangle = 0$ for all $x \in V$, then $S = T_0$, that is $S(x) = 0$ for all $x \in V$.

**Note:** $R_{\pi/2} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a normal operator where $\langle x, R_{\pi/2}(x) \rangle = 0$ for all $x \in \mathbb{R}^2$.

**Proof of Lemma:** Pick $\{v_1, \ldots, v_n\}$ an orthonormal basis of eigenvectors for $S$. If $S(v_i) = \lambda_i v_i$.
Then $0 = \langle S(v_i), v_i \rangle = \langle \lambda_i v_i, v_i \rangle = \lambda_i \| v_i \|^2$
and so all $\lambda_i = 0$. Thus $S(v_i) = 0$ for all $i$ and
hence $S = T_0$.

\[ \square \]

**Proof of Thm:**

(a) \Rightarrow (b): By last time, as $T^* T$ is self-adjoint,
so is $S = T^* T - I_V$. For any $x \in V$ we have
\[
\langle S(x), x \rangle = \langle T^* T(x), x \rangle - \langle I_V(x), x \rangle
= \langle T(x), T(x) \rangle - \langle x, x \rangle
= 0 \text{ since } T \text{ is an isometry.}
\]

By the lemma, $S = T_0$ and so $T^* T = I_V$.

Then $T^* = T^{-1}$ and so $T \circ T^* = T_0 T^{-1} = I_V$
as well, proving (b).

(b) \Rightarrow (c): Suppose $\beta = \{ u_1, \ldots, u_n \}$, and
set $w_i = T(u_i)$. Then
\[
\langle w_i, w_j \rangle = \langle T(u_i), T(u_j) \rangle = \langle u_i, T^* T(u_j) \rangle
= \langle u_i, I_V(u_j) \rangle = \langle u_i, u_j \rangle
\]
Hence $\{ w_1, \ldots, w_n \}$ is an orthonormal basis, as
needed for (c).
(c) \implies (d): Clear.

(d) \implies (e): Suppose \( \beta = \{ u_1, \ldots, u_n \} \), \( w_i = T(u_i) \), with \( \beta \) and \( \gamma = \{ w_1, \ldots, w_n \} \) orthonormal. Write

\[ x = a_1 u_1 + \ldots + a_n u_n. \]

Then \( \| x \|^2 = \langle x, x \rangle \)

\[ = \sum_{i,j} a_i \overline{a_j} \langle u_i, u_j \rangle = \sum_{i=1}^n |a_i|^2 \quad \text{and} \]

Similarly, \( \| T(x) \|^2 = \| \sum a_i w_i \|^2 = \sum_{i=1}^n |a_i|^2 \).

Thus \( \| T(x) \| = \| x \| \) as needed.

(e) \implies (a): By expanding \( \langle x+y, x+y \rangle \), get that

\[ \text{real part} \]

\[ \text{Im} (\langle x, y \rangle) = \frac{1}{2} \left( \| x+y \|^2 - \| x \|^2 - \| y \|^2 \right) \]

and if \( F = \mathbb{C} \) then

\[ \text{Im} (\langle x, y \rangle) = -\frac{1}{2} \left( \| x+y \|^2 - \| x \|^2 - \| y \|^2 \right) \]

Thus \( \langle , \rangle \) is actually determined by the assoc. norm \( \| \cdot \| \), and so as \( T \) preserves the former it preserves the latter. \( \square \)
If a linear op $T$ of $V$ is an isometry and $\beta$ is orthonormal, then by (b) we have

$$I_n = [I_V]_\beta = [T^* \circ T] = [T^*]_\beta [T]_\beta$$

$$= ([T]_\beta)^* [T]_\beta$$

**Def:** A square matrix $A$ is **unitary** if $A^* A = I$. It is **orthogonal** if $A^t A = I$.

So the matrix of an isometry is always unitary, and when $V = IR$ it is also orthogonal.

**Thm:** Suppose $A \in M_{n \times n}(IR)$ is orthogonal.

Then $L_A$ is an isometry of $(IR^n, \text{dot})$.

**Note:** Analogous true for $A \in M_{n \times n}(C)$ that are unitary.

**Proof:** Let $\beta = \{e_1, \ldots, e_n\}$ and set $a_i = A e_i$ = $i^{th}$ col of $A$. Then $A^t A = \begin{pmatrix} -a_1^2 & \cdots & -a_n^2 \\ a_1 \cdots a_n \end{pmatrix} (a_1 \cdots a_n)$

$$= I$$

where $G_{ij} = \langle a_i, a_j \rangle$. 

Thus \( \gamma = \{a_1, \ldots, a_n\} \) and \( \gamma' = \{v_A(\beta)\} \) is an orthonormal basis for \( \mathbb{R}^n \), and so \( L_A \) is an isometry by the first thm.

Cor: \( A \in M_{n \times n}(\mathbb{R}) \). The following are equivalent

i) \( A \) is orthogonal.

ii) \( A^t = A^{-1} \)

iii) The columns of \( A \) are an orthonormal basis for \( \mathbb{R}^n \)

iv) The rows of \( A \) ________ " ________

Pf: Exercise.

Restated Thm: Suppose \( A \in M_{n \times n}(\mathbb{R}) \) is symmetric. Then there is an orthogonal matrix \( Q \) with \( Q^tAQ = Q^{-1}AQ \) diagonal.