Lecture 37: Diagonalizing self-adjoint operators. (§6.4)

Last time: A linear operator $T$ on an inner product space $V$ is self-adjoint when $T = T^*$. A square matrix is self-adjoint if $A = A^*$.

Synonyms: Hermitian (field = $\mathbb{C}$), symmetric (field = $\mathbb{R}$).

Ex: Suppose $W$ is a subspace of a finite dimensional inner product space $V$. Then orthogonal projection $T = \text{proj}_W : V \to V$ is self-adjoint.

Proof: Suppose $y_1, y_2 \in V$ and write $y_i = w_i + z_i$ where $w_i = T(y_i) \in W$ and $z_i \in W^\perp$. Then

$$\langle T(y_1), y_2 \rangle = \langle w_1, w_2 + z_2 \rangle =$$

$$\langle w_1, w_2 \rangle + \langle w_1, z_2 \rangle = \langle w_1, w_2 \rangle$$

$$= \langle w_1 + z_1, w_2 \rangle = \langle y_1, T(y_2) \rangle.$$
Thus $T$ is self-adjoint.

\[ R(\text{proj}_W) = W \text{ and } N(\text{proj}_W) = W^+. \]

In particular, $R^+ = N$. On HW, will show this is a general property of normal operators. On to today's goal...

Thm: Suppose $T$ is a self-adjoint operator on a finite-dim'l inner product space $V$. Then $V$ has an orthonormal basis $\beta$ consisting of eigenvectors for $T$.

Cor: If $A \in \text{M}_{n \times n}(\mathbb{R} \text{ or } \mathbb{C})$ then $A$ is diagonalizable.

Lemma (last time) Any eigenvalue of a self-adjoint $T$ is real.

Lemma: Any self-adjoint operator $T$ on $V \neq \{0\}$ has an eigenvector.
Proof: Let $\beta$ be an orthonormal basis for $V$ and set $A = [T]_\beta$. Note it suffices to show $A$ has an eigenvector. If the field is $\mathbb{C}$, we're done as all matrices in $M_{n \times n}(\mathbb{C})$ have an eigenvector (reason: the char poly must have at least one root). So assume the field is $\mathbb{R}$, in which case $A = A^* = A^t$. It suffices to show that the char poly $f(t)$ has a real root. Now $f(t)$ has at least one root $\lambda$ in $\mathbb{C}$, which means that $L_A: \mathbb{C}^n \to \mathbb{C}^n$ has an eigenvector with eigenvalue $\lambda$. As $L_A$ is self-adjoint (as $A^* = A$), we have $\lambda \in \mathbb{R}$ by the previous lemma. So $f(t)$ has a real root as needed.  \[\square\]
Proof of theorem: We induct on $\dim(V)$. If $\dim(V) = 1$, the lemma gives eigenvector $v$ for $T$. As $v \neq 0$, we have $\beta = \{ \frac{v}{\|v\|} \}$ is the orthonormal basis of eigenvectors we seek.

Now assume true when $\dim(V) \leq n-1$, and consider $V$ of $\dim n$. Let $v_i$ be a unit eigenvector of $T$ given by the lemma. Consider $W = \{ v_1 \}^\perp$.

Claim: $T(W) \subseteq W$

Reason: Suppose $w \in W$. Then $\langle T(w), v_i \rangle = \langle w, T^*(v_i) \rangle = \langle w, \lambda_i v_i \rangle = \lambda_i \langle w, v_i \rangle = 0$ since $w \in \{ v_i \}^\perp$. Thus $T(w)$ is also in $W$. 

Now $W$ is also an inner product space (with the inner prod. inherited from $V$), and the restricted linear op $T_W: W \to W$ is still self-adjoint. By induction, there is an orthonormal basis $\{v_2, \ldots, v_n\}$ of $W$ consisting of eigenvectors of $T_W$. Then

$$\beta = \{v_1, v_2, \ldots, v_n\}$$

is an orthonormal set (since $\langle v_i, v_i \rangle = 0$ for $i > 1$ as $v_i \in \ker T^2$) of eigenvectors of $T$.

As any orthonormal set is linearly indep., and $\dim V = n$, we have shown $\beta$ is a basis as needed.

Next time: Orthogonal and unitary operators, (§ 6.5)