Thm: \( A \in M_{n \times n}(\mathbb{R}) \) is diagonalizable if and only if there is a basis of \( \mathbb{R}^n \) consisting of eigenvectors for \( A \).

Def: If \( \lambda \) is an eigenvalue for \( A \in M_{n \times n}(\mathbb{R}) \),
then \( E_\lambda = \{ v \in \mathbb{R}^n \mid Av = \lambda v \} = \mathcal{N}(A - \lambda I_n) \)
is called the eigenspace of \( A \) corresponding to \( \lambda \).

Today's Moral: There are two things that can prevent diagonalization:

1. Too few eigenvalues: \( A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \)
   
   Char poly = \( \det(A - tI_2) = \det \begin{pmatrix} -t & -1 \\ 1 & -t \end{pmatrix} = t^2 + 1 \).
   
   This has no roots in \( \mathbb{R} \), so \( A \) has no eigenvalues/eigenvectors at all!
Note: Geometrically, \( L_A : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is rotation by \( \pi/2 \), so clearly no eigenvectors.

Potential fix: Enlarge field to \( \mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\} \)
where \( i^2 = -1 \). See HW for more.

2. Too few eigenvectors: \( B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Then char poly = \( (t-1)^2 \) and so all eigenvectors live in:

\[
E_1 = \mathcal{N}(B-I_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) = \{ (t, 0) \mid t \in \mathbb{R}^2 \}
\]

So any eigenvector is a scalar mult of any other \( \Rightarrow \) Can't have a basis of eigenvectors, so not diagonalizable.
**Def:** A polynomial $f(t)$ with coefficients in a field $\mathbb{F}$ splits (or splits completely) over $\mathbb{F}$ if
\[ f(t) = c(t-a_1)(t-a_2) \ldots (t-a_d) \]
where $c, a_1, \ldots, a_d \in \mathbb{F}$ and $d = \deg(f)$.

**Fundamental Theorem of Algebra:** Every polynomial with coefficients in $\mathbb{C}$ splits completely over $\mathbb{C}$.

**Thm:** If $A \in M_{n \times n}(\mathbb{R})$ is diagonalizable, then its char. poly. splits completely over $\mathbb{R}$.

**Proof:** As $A$ is similar to a diagonal matrix, there is a $Q \in M_{n \times n}(\mathbb{R})$ with $Q^{-1}AQ = D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$. Then \( \det(A-tI_n) = \) \( \det(QDQ^{-1} - Q(tI_n)Q^{-1}) = \)
\[ \det(Q(D-tI_n)) = \det \begin{pmatrix} \lambda_1-t & 0 & \cdots & 0 \\ 0 & \lambda_2-t & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n-t \end{pmatrix} \]
\[= (\lambda_1 - t)(\lambda_2 - t) \ldots (\lambda_n - t)\]

and so the char poly of A splits completely.

[This gives a necessary condition for A to be diagonalizable. Is it also sufficient? No!]

\textbf{Def}: Suppose \( \lambda \) is an eigenvalue of A. The \underline{algebraic multiplicity} of \( \lambda \) is the number of times that \((t - \lambda)\) divides the char poly of A. The \underline{geometric multiplicity} of \( \lambda \) is \( \dim(\mathbb{E}_\lambda) \).

\textbf{Thm}: For each \( \lambda \), \((\text{geom multi}) \leq (\text{alg multi})\)

\textbf{Thm}: A matrix \( A \in M_{n \times n}(\mathbb{R}) \) is diagonalizable if and only if

a) char poly of A splits completely.

b) For every eigenvalue \( \lambda \), \((\text{geom multi}) = (\text{alg multi})\).
Cor: If the char poly of $A \in \text{Mat}_{n \times n} (\mathbb{R})$ has $n$ distinct roots in $\mathbb{R}$ then $A$ is diagonalizable.

Proof of Cor: If the char poly has $n$ distinct roots and degree $n$, then it splits completely. Moreover every eigenvalue has alg. mult. 1. As the geom mult of any $\lambda$ is $\geq 1$ we thus have that all alg and geom. mult. agree and so the theorem applies. \( \square \)

Proof of Thm about multiplicities: Let $\\{v_1, \ldots, v_k\}$ be a basis for $E_\lambda$. Enlarge this to a basis $\beta = \{v_1, v_2, \ldots, v_n\}$ for $\mathbb{R}^n$. Then

\[
\begin{bmatrix}
L_A
\end{bmatrix}_{\beta} = \begin{pmatrix}
\lambda & 0 & \cdots & 0 \\
0 & \lambda & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda
\end{pmatrix}_k \quad \text{and} \quad L_A(v_i) = \lambda v_i \quad \text{for} \quad 1 \leq i \leq k.
\]

Setting $Q = [I_{\mathbb{R}^n}]_{\beta^{-1}}$ (i.e. $Q^{-1} A Q = \lambda I$), have
\([L_A]_\beta = Q^{-1}AQ\). As before, we see similar matrices have the same char. poly, and so

\[
\text{Char poly } A = \det \left( [L_A]_\beta - t I_n \right)
\]

\[
= \det \begin{pmatrix}
\lambda - t & 0 & \cdots & 0 \\
0 & \lambda - t & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda - t
\end{pmatrix}
\]

\[
= (\lambda - t)^k \det (?)
\]

Thus \( \text{alg mult } \lambda \) \(\geq k \) as required.\]