
Important note: This assignment is due on Wednesday not Friday.

More important note: This is the last homework assignment of the semester!

Most important note: There will be a combined final exam for sections B13 and C13 of Math 416, which will be held on Friday, May 6 from 1:30-4:30 in Psychology 23. Please notify me immediately if you have another exam in that timeslot.

Webpage: [http://dunfield.info/416](http://dunfield.info/416)

Office hours: Here is my schedule for the rest of the semester:
- Friday, April 29: 11am–noon.
- Monday, May 2: 3:30–4:30pm.
- Tuesday, May 3: 3:30–4:30pm.
- Thursday, May 5: noon–1:30pm, 3:30–4:30pm.
- Friday, May 6: 9–10am.

Problems:

1. Suppose $V$ is a finite-dimensional inner product space and $T$ a linear operator on $V$. Prove that $R(T^*)^\perp = N(T)$ and $R(T^*) = N(T)^\perp$.

2. Let $T$ be a normal operator on a finite-dimensional inner product space $V$.
   
   (a) Prove that $N(T) = N(T^*)$ and $R(T) = R(T^*)$.
   
   (b) Prove that the subspaces $N(T)$ and $R(T)$ are orthogonal.
   
   (c) Give an example of a (non-normal) linear operator $S$ where $N(S) \neq N(S^*)$ and $R(S) \neq R(S^*)$.

   Hint: Problem 1 is your friend here.

3. A matrix $A \in M_{n \times n}(\mathbb{R})$ is Gramian if there is a $B \in M_{n \times n}(\mathbb{R})$ such that $A = B^tB$. Prove that $A$ is Gramian if and only if $A$ is symmetric and all of its eigenvalues are non-negative.

   Hint: For $(\Leftarrow)$, note that $A$ is diagonalizable via an orthonormal basis $\{u_1, \ldots, u_n\}$ where $u_i$ is an eigenvector of $A$ with eigenvalue $\lambda_i$. Consider the linear operator $T$ on $\mathbb{R}^n$ where $T(u_i) = \sqrt{\lambda_i}u_i$. Now take $B = [T]_{\text{std}}$ and check that $A = B^tB$.

4. Section 6.5 of [FIS], Problem 11.

5. Section 6.5 of [FIS], Problem 17.

6. Section 6.5 of [FIS], Problem 24.

7. Suppose $A \in M_{3 \times 3}(\mathbb{R})$ is an orthogonal matrix with $\det(A) = 1$. (Recall from a prior assignment that any orthogonal matrix has determinant $\pm 1$.) In this problem, you will show $L_A$ is rotation about a line $W$ in $\mathbb{R}^3$, where $W$ passes through the origin.

   (a) First, show that any (real) eigenvalue of $A$ must be $\pm 1$. 

(b) Note that $A$ has at least one eigenvalue since its characteristic polynomial $f(t)$ has odd degree and hence at least one real root $\lambda$. In this step, you’ll show that 1 is always an eigenvalue. If instead $\lambda = -1$, then $f(t) = (-1 - t)(t^2 + bt + c)$ for some $b, c \in \mathbb{R}$. Use that $\det(A) = 1$ to prove that $c < 0$ and hence by the quadratic formula that $f(t)$ splits completely over $\mathbb{R}$. Now show that the eigenvalues of $A$ are $-1$ and 1, with algebraic multiplicies 2 and 1 respectively.

(c) Let $v_1$ be an eigenvector for $A$ with eigenvalue 1, and set $W = \text{span}\{v_1\}$. Prove that $L_A$ preserves $W^\perp$ and acts on it by an orthogonal transformation.

(d) Use Theorem 6.23 of the text to argue that the action of $L_A$ on $W^\perp$ is by a rotation. Hint: If instead the restriction was a reflection, find a basis of $\mathbb{R}^3$ consisting of eigenvectors for $A$ which shows instead that $\det(A) = -1$.

8. Suppose $v_1, \ldots, v_n$ are vectors in $\mathbb{R}^n$ and let $P$ be the parallelepiped spanned by them. Consider the matrix $G \in M_{n \times n}(\mathbb{R})$ where $G_{ij} = \langle v_i, v_j \rangle$. (As usual, the inner product here is just the ordinary dot product.)

(a) Show that $G$ is Gramian.

(b) Show that $\det(G) \geq 0$.

(c) Show that the unsigned volume of $P$ is $\sqrt{\det(G)}$.

In fact, $G$ is usually called the Gram matrix of these vectors.