Lecture 6: Level sets in $3^d$ (§14.1) and quadric surfaces (§12.6); review of limits (§14.2)

Last time: $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = x^2 - y^2$

Graph

For $f : \mathbb{R}^3 \to \mathbb{R}$, can't draw the graph (it's in $\mathbb{R}^4$) but can still look at level sets.

[Did $f(x, y, z) = x^2 + y^2 + z^2$ last time.]

Ex: $f(x, y, z) = x^2 + y^2 - z^2$

First, in the $xz$-plane $f(x, 0, z) = x^2 - z^2$

so the level sets there match the above picture.
Another important tool for drawing graphs and level sets is:

**Symmetry:** As $r^2 = x^2 + y^2$, we can write $f(x, y, z) = r^2 - z^2$. Thus, each level set is rotationally symmetric about the $z$-axis.

These level sets are examples of quadric surfaces.
Conic sections: Solution is $\mathbb{R}^2$ of

$$A x^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$ 

- circle
- ellipse
- parabola
- hyperbola

Quadratic surfaces in $\mathbb{R}^3$ (§12.6 and upcoming worksheet)

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Gx + Hy + Iz + J = 0$$

Ex: Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Elliptic paraboloid:

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Hyperbolic paraboloid:

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$
The other quadric surfaces are the (double) cone and the hyperboloids of one and two sheets. You'll learn more about these in section and on the HW.

**Limits (§14.2)**

To talk about derivatives of functions of several variables, we need to understand limits:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

There are different perspectives on limits; I'll focus on them as a way of estimating/controlling error.

Suppose we want to fabricate a square with area $1 \text{m}^2$ but it comes back with sides of length $1+h$.

$$\text{area} = (1+h)^2 - 1 = h^2 + 2h$$

Q: If we want this error to be $\frac{1}{10}$, to what tolerance do we need to make the square?
Consider \( E : \mathbb{R} \to \mathbb{R} \) (think "error function"). We say \( \lim_{h \to 0} E(h) = 0 \) if given \( \varepsilon > 0 \) we can always find \( \delta > 0 \) so that whenever \( 0 < |h| < \delta \) we have \( |E(h)| < \varepsilon \).

\[
\begin{array}{c}
E(h) \\
\end{array}
\]

Ex: \( E(h) = h^2 \)

[View as challenge-response process.]

1st Challenge: \( \varepsilon = \frac{1}{10} \). Take \( \delta = \frac{1}{4} \). If \( |h| < \delta = \frac{1}{4} \) then \( |E(h)| = |h^2| = |h|^2 < \frac{1}{16} < \frac{1}{10} \).

2nd Challenge: \( \varepsilon = \frac{1}{100} \) \( \delta = \) Audience response

3rd Challenge: \( \varepsilon = \frac{1}{10000} \) \( \delta = \) —— “” ——
Claim: \( \lim_{h \to 0} h^2 = 0 \).

Reason: If you give me \( \varepsilon > 0 \), I'll take \( \delta = \sqrt{\varepsilon} \).

Then if \( |h| < \delta \) we have \( |h|^2 = |h|^2 < \delta^2 = \varepsilon \), as desired.

Ex: \( E(h) = h^2 + 2h \) \( \lim_{h \to 0} h^2 + 2h = 0 \)

by "limit laws" from Calc I.

Given \( \varepsilon = \frac{1}{10} \), take \( \delta = \frac{1}{100} \). If \( |h| < \delta \), then
\[
|h^2 + 2h| \leq |h|^2 + 2|h| < \left( \frac{1}{100} \right)^2 + \frac{2}{100}
\]
\[
< \frac{3}{100} < \frac{1}{10} = \varepsilon.
\]

Note: In general, say \( \lim_{x \to a} f(x) = c \)

if \( f(a + h) = c + E(h) \)

where \( \lim_{h \to 0} E(h) = 0 \).