Lecture 5: Functions of Several Variables (§14.1)

Function of one variable

\[ f(x) = x^2 \]

Function of two variables

\[ f(x, y) = x^2 - xy \]

In general, will consider

\[ f: \mathbb{R}^n \rightarrow \mathbb{R}^m \]

Ex: 1) Temperature in this room \( T: \mathbb{R}^3 \rightarrow \mathbb{R} \)

2) Parameterized curve in the plane \( r: \mathbb{R} \rightarrow \mathbb{R}^2 \)

\[ t = \text{time} \]

\[ 0 \rightarrow r(1) \rightarrow r(0) \rightarrow r(-1) \]
Today: Functions $\mathbb{R}^2 \to \mathbb{R}$.

**Graphs:** For one var $f: \mathbb{R} \to \mathbb{R}$, such as $f(x) = x^2$, have:

Consider $f: \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = x^2 + y^2$.

Its graph is $\{(x,y,f(x,y))\}$ in $\mathbb{R}^3$.

and is shown on the next page.
How to figure out: Intersect graph with planes.

What is over the $x$-axis? (or $y$-axis,)

What is the intersection with $\{ z = c \}$?

Same as finding all $(x, y)$ with $f(x, y) = c$, that is, $x^2 + y^2 = c$.
Other tools: symmetry, computers.

Ex: \( f: \mathbb{R}^2 \to \mathbb{R} \) with \( f(x, y) = x^2 - y^2 \)

- Over x-axis: \( f(x, 0) = x^2 \)
- Over y-axis: \( f(0, y) = -y^2 \) \( \{ \text{Both parabolas} \} \)

Intersections w/ horizontal planes:

\( Z = 0 \):
\[ x^2 - y^2 = 0 \]
\( \iff \)
\[ x^2 = y^2 \]
\( \iff \)
\[ x = \pm y \]

\( Z = -1 \):
\[ x^2 - y^2 = -1 \]
\( \iff \)
\[ y^2 = x^2 + 1 \]
\( \iff \)
\[ y = \pm \sqrt{x^2 + 1} \]

\( Z = +1 \):
\[ x^2 - y^2 = 1 \]
\( \iff \)
\[ x = \pm \sqrt{y^2 + 1} \]

See next page for plot.
Each of these curves is a level set, like a contour line on a map. Put together, we have
In other words, the graph is a saddle:

Another example: \( f: \mathbb{R}^2 \rightarrow \mathbb{R} \), \( f(x,y) = x + y + 1 \)

Graph is \( z = f(x,y) \), that is
\[ x + y - z + 1 = 0 \]
and hence is a plane

Now, let's try more variables!
Ex: \( f: \mathbb{R}^3 \to \mathbb{R} \) \( f(x, y, z) = x^2 + y^2 + z^2 \)

No graph [well, there is one in \( \mathbb{R}^4 \)]

but we still have level sets.

\[ f = 1 \iff x^2 + y^2 + z^2 = 1 \]

Ex: Hopf fibration \( f: \mathbb{R}^3 \to \mathbb{R}^3 \)

has image contained in \( \mathbb{S}^2 \). Level sets are mostly circles. [Show video.]