Lecture 3: Projection via the dot product (§12.3).

Lines and planes in \( \mathbb{R}^3 \) (§12.5).

Last time: \( \vec{V} = (v_1, v_2, v_3) \) \( \vec{W} = (w_1, w_2, w_3) \)
\[ \vec{V} \cdot \vec{W} = v_1 w_1 + v_2 w_2 + v_3 w_3 \]

Key: \( \vec{V} \cdot \vec{W} = |\vec{V}| |\vec{W}| \cos \theta \)

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Projection:

\[ \text{Proj}_\vec{V} \vec{W} = \text{component of } \vec{W} \text{ along } \vec{V}. \]

\[ = \text{scalar mult. of } \vec{V} \text{ closest to } \vec{W}. \]

Note: \[ |\text{Proj}_\vec{V} \vec{W}| = |\vec{W}| \cos \theta = \frac{\vec{V} \cdot \vec{W}}{|\vec{V}|} \]

So if \( \hat{u} = \text{unit vector pointing in same direction as } \vec{V} \)

\[ \text{Proj}_\vec{V} \vec{W} = |\vec{W}| \cos \theta \hat{u} = \left( \frac{\vec{V} \cdot \vec{W}}{|\vec{V}|} \right) \left( \frac{\vec{V}}{|\vec{V}|} \right) \]

\[ = \frac{\vec{V} \cdot \vec{W}}{|\vec{V}|^2} \vec{V} \]}
\[ \text{Work} = (\text{force}) \times (\text{distance}) \]

\[ F \rightarrow \text{distance} \rightarrow \text{work} \]

\[ \vec{F}_u \rightarrow \vec{d} \rightarrow \vec{F} \]

\[ W = |\vec{F}_u| \| \vec{d} \| = |\text{proj}_d \vec{F}| \| \vec{d} \| = (\frac{\vec{d} \cdot \vec{F}}{\| \vec{d} \|}) \| \vec{d} \| = \vec{d} \cdot \vec{F} \]

So \[ W = \vec{F} \cdot \vec{d} \]

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**Regression:**

\[ n \text{ samples } (x_i, y_i) \]

Roughly, \[ y_i = c x_i \]

What is \( c \)?

Consider

\[ \hat{x} = (x_1, x_2, \ldots, x_n) \text{ in } \mathbb{R}^n \]

\[ \hat{y} = (y_1, y_2, \ldots, y_n) \]

If \( y_i = c x_i \) exactly for all \( i \), then \( \hat{y} = c \hat{x} \)
In general, model has more parameters, and projection is onto a plane or a higher dimensional analog. See Math 415/416.

Planes in $\mathbb{R}^3$:

A plane in $\mathbb{R}^3$ can be specified by a point $P_0 = (x_0, y_0, z_0)$ and a normal vector $\hat{n}$ that meets the plane at right angles. Another point $P = (x, y, z)$ is in our plane exactly when $\hat{n}$ and $\hat{V}$ are perpendicular (orthogonal), i.e. $\hat{n} \cdot \hat{V} = 0$. So, the best fit is given by

$$c = \frac{\hat{x} \cdot \hat{y}}{|\hat{x}|^2}$$
Thus $P$ is in the plane when

$$0 = (a, b, c) \cdot (x - x_0, y - y_0, z - z_0)$$
$$= a(x - x_0) + b(y - y_0) + c(z - z_0)$$
$$= ax + by + cz + d \text{ where } d = -(ax_0 + by_0 + cz_0)$$

Conversely,

$$ax + by + cz + d = 0$$
defines a plane, unless $a = b = c = 0$.

[Normal vectors will be key concept later in this course. For now, will use to solve geometric problems about planes.

Ex: $P_1 = \text{Plane given } x + y + z - 1 = 0$

Any plane is determined by 3 pts.

Normal vector is

$$\vec{m}_4 = (1, 1, 1)$$
$P_2 = \{ z = 1 \}$

**Q1:** What is the intersection of $P_1$ and $P_2$?

**Q2:** What is the angle between them?

**A1:** A line:

Points on $L$ satisfy $x + y + z = 1$ and $z = 1$

Two easy solutions:

Get $\vec{V} = \vec{r}_1 - \vec{r}_0$

$= (1, -1, 1) - (0, 0, 1)$

$= (1, -1, 0)$

Thus every point on $L$ has the form

$\vec{r}(t) = \vec{r}_0 + t \vec{V}$
because of:

Thus the line $L$ is parameterized by

$$\vec{r}(t) = (0, 0, 1) + t(1,-1,0)$$

$$= (0,0,1) + (t,-t,0)$$

$$= (t,-t,0)$$

(This is just like Problem 3(e) from yesterday's worksheet.)

A2: Angle between the planes is the same as the angles between the normals.

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1||\vec{n}_2|} = \frac{1}{\sqrt{3} \cdot 1} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta \approx 54.7^\circ$$