The epsilon-delta definition of the limit

We first define what it means for a function to have “limit 0 at 0”.

Definition. Let the function \( E \) be defined on an open interval about 0, except possibly at 0. We say that \( \lim_{h \to 0} E(h) = 0 \) if

for every challenge number \( \epsilon > 0 \),
there is a response number \( \delta > 0 \) so that
if \( 0 < |h| < \delta \),
then \( |E(h)| < \epsilon \).

Example: \( E(h) = h^2 \) has limit 0 at 0:

Given an arbitrary \( \epsilon > 0 \),
we can choose \( \delta = \sqrt{\epsilon} \). Then
if \( 0 < |h| < \delta \),
\[ |E(h)| = |h^2| < \delta^2 = \epsilon. \]

Exercise: Prove these facts using epsilon-delta arguments:

- If functions \( E_1 \) and \( E_2 \) both have limit 0 at 0, so does their sum.
- If functions \( E_1 \) and \( E_2 \) both have limit 0 at 0, so does their product.
- If function \( E_1 \) has limit 0 at 0, and \( |E_2(h)| < |E_1(h)| \) for all \( h \), then \( E_2 \) has limit 0 at 0.
- If \( |g(h)| < M \) all \( h \), and \( E \) has limit 0 at 0, then the function \( gE \) has limit 0 at 0.

For a more general function \( F \) defined on an open interval about \( a \), except possibly at \( a \), we say

\[ \lim_{x \to a} F(x) = L \]

if the “error” function \( E(h) = F(a + h) - L \) has limit 0 at 0.

For example, suppose \( F(x) = x^2 \), and we want to show \( \lim_{x \to 2} F(x) = 4 \).

We let \( E(h) = F(2 + h) - 4 = (2 + h)^2 - 4 = 4h + h^2 \), and show \( E(h) \) has limit 0 at 0.

If \( \epsilon = \frac{1}{10} \), we can choose \( \delta = \frac{1}{100} \), since then \( |E(h)| = |h^2 + 4h| \leq |h^2| + |4h| < \frac{1}{10000} + \frac{4}{100} < \frac{5}{100} < \frac{1}{10} \).

The case of an arbitrary \( \epsilon \) is harder. Instead of finding \( \delta \) directly, we could show \( E_1(h) = h^2 \) and \( E_2(h) = 4h \) both have limit 0 at 0, and then use the fact that their sum also has limit 0 at 0.