Goal Thm: $G$ finite gp. Then $\exists$ a Galois extension $K/C(t)$ with group $G$.

Plan: 1. Find a curve $V$ in $\mathbb{P}^n_C$ on which $G$ acts by symmetries, so that $V/G = \mathbb{P}^1_C$.
2. $G$ is now a subgp of $\text{Aut}(K=C(V))$.
3. $K_G = C(V)_G = C(V/G) = C(\mathbb{P}^1_C) = C(t)$.

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Given a group $G$, let's make it act on some geometric object.

**Def:** Let $S$ be a generating set for $G$. The Cayley graph $\Gamma(G,S)$ has

1. a vertex $V_g$ for each $g \in G$
2. an edge labeled $s$ from $V_g$ to $V_{gs}$ $\forall g \in G, s \in S$.

**Ex:** $G = C_2 \times C_2 = \{1, \tau, \sigma, \tau \sigma\}$

$S = \{\tau, \sigma\}$
Ex: \( G = C_n \), \( S = \{ \text{gen } \sigma \} \)

\[
\begin{align*}
\text{Ex: } S_3 & = \{ 1, (12), (13), (23), \\
& (123), (132) \} \\
S & = \{ a = (12), \ b = (123) \}
\end{align*}
\]

g joined to gs:

\[(12)(123) = (1)(23)\]

Q: What is \(abab^{-1}ab\)?

A: \(a = (12)\)

For any \((G, S)\), the group \(G\) acts on \(\Gamma(G, S)\) by \(g \cdot V_h = V_{gh}\). This respects the edges, since an "s" edge joins \(V_h \rightarrow V_{hs}\) and so there is also an "s" edge from \(g \cdot V_h = V_{gh}\) to \(g \cdot V_{hs} = V_{ghs}\).
Aside: Can also do for infinite groups, leading to geometric group theory:

- Certain families of Cayley graphs are expanders:

\[ G = \text{PSL}_2 \mathbb{F}_p \quad S = \{ (0,1), (1,0) \} \]

In the main example:

a acts on \( \Gamma \) by rotation by \( \pi \):

\[ (123) \rightarrow (132) \]

b acts on \( \Gamma \) by rotation by \( 2\pi/3 \):

\[ (12)(132) = (13) \]

\[ (12)(123) = (23) \]
What is $\Gamma/G$?  A: $b \circlearrowleft \rightarrow a$.

As we want $G$ to act on a surface, thicken $\Gamma/G$ to

and correspondingly thicken $\Pi$ to

For each boundary circle, add a disc.

$\Gamma/G$ becomes

$X = \_ \_ = P_c'$
and $\Gamma'$ becomes $\gamma = \bigcirc$ as well.

The action of $G$ on $\Gamma'$ gives an action of $G$ on $\gamma$.

\[
\begin{align*}
(13)\quad & b \\
2\pi/3\quad & (23)\text{ rotates by } \pi \\
\pi/3\quad & a
\end{align*}
\]

\[
S_3 = \text{orientation-preserving symmetries of the bipyramid.}
\]

There $p: \gamma \to \gamma/G = X$ extending $\Gamma \to \Gamma/G$.

First, note that $\Gamma \to \Gamma/G$ is locally 1-1 (a homeomorphism). The same is true for $p: \gamma \to X$, except at the 8 points...
That are fixed by some elt of \( G \), which are the centers of the added discs.

At these points, the quotient map \( p \) looks like

\[
\begin{align*}
\mathbb{Z} &\rightarrow \mathbb{Z}^2 \\
\mathbb{Z} &\rightarrow \mathbb{Z}^3
\end{align*}
\]

So \( p: Y \rightarrow X \) is a branched covering map which looks locally like a polynomial.

Next time: We will invoke the Riemann existence theorem to turn this into an honest rat'l map \( Y \rightarrow X \), giving an extension \( K/\mathbb{C}(t) \) with
Galois group $S_3$.

Note: The construction of $p: Y \to X = P^1_c$ from $\Gamma_1(6,5)$ is completely general.

It's the Riemann existence theorem that's hard...