Lecture 8: Field extensions II

Last time:

- $K/F$, a field extension means $F \subseteq K$
- $[K:F] =$ dim of $F$ as a $K$-vector space.
- $p(x)$ irreducible poly in $F[x]$ can form a field $K = F[x]/(p(x))$ polynomials in $F[x]$ of degree $< \deg p$.

$\Rightarrow [K:F] = \deg p$.

Think of $K$ as adding a root of $p(x)$ to $F$. Explicitly, set $\theta = x + (p(x))$. Then $p(\theta) = p(x) + (p(x)) = 0$.

A $F$ basis of $K$ is $1, \theta, \theta^2, \ldots, \theta^n$ where $n = \deg p - 1$.

Ex: $F = \mathbb{R}$, $p = x^2 + 1$ $F = \mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$

\[ \begin{array}{c c c}
1 & \rightarrow & 1 \\
\theta & \rightarrow & i (\alpha - i)
\end{array} \]

Notation: $\alpha_1, \ldots, \alpha_n \in K$ with $F \subseteq K$. Then $K(\alpha_1, \alpha_2, \ldots, \alpha_n) =$ field gen by $F$ and the $\alpha_i$ [i.e. the smallest subfield of $K$ which contains them.]

Ex: $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{2}, \sqrt{5})$. Here the large field is $\mathbb{C}$. 

Simple extension: $K = F(\alpha)$ for some $\alpha$ in $K$

\[ \text{primitive element} \]

$\mathbb{Q}(\sqrt{2}, \sqrt{5}) / \mathbb{Q}$ is simple as $\mathbb{Q}(\sqrt{2}, \sqrt{5}) = \mathbb{Q}(\sqrt{2} + \sqrt{5})$

Since $\sqrt{2} = \frac{1}{6}(\alpha^3 - 11\alpha)$.

Fact: [Will show] Any $K/F$ with $[K:F] < \infty$ and $\text{ch}(F) = 0$ is simple.

Thm: $p(x) \in F[x]$ irreducible. Suppose $K$ is a simple extension of $F$ with primitive elt $\alpha$. If $p(\alpha) = 0$, then
\[ L = F[x]/(p(x)) \cong K \]

Pf: Consider $\phi: L \to K$ given by $g(x) + (p(x)) \mapsto g(\alpha)$; makes sense because $f(\alpha) = 0$ if $f \in (p(x))$. Note $\phi$ is a ring homom. by the basic ring axioms.

Lemma: $\psi: L \to K$ a ring homom. of fields. Then either $\psi(L) = 0$ or $\psi$ is 1-1.

Reason: $\ker \psi = \{ \psi(x) = 0 \mid x \in L \}$ is an ideal, hence either $0$ or $L$ as every non-zero elt of $L$ is a unit.
Our $\phi$ is not trivial, since $\phi|_{\text{const}}$ is an isomorphism to $F$. So $\phi$ is 1-1, and moreover $\phi$ is onto since its image contains $F$ and $\alpha$. So $\phi$ is an isom. □

**Thm.** Suppose $K = F(\alpha)$ with $[K : F] = n < \infty$
Then $\exists$ an irreducible poly $p(x) \in F[x]$ with $p(\alpha) = 0$.
Thus $K \cong F[x]/(p(x))$.

**Ex:** $\mathbb{Q}(\sqrt{2}) \cong \mathbb{Q}[x]/(x^2 - 2)$

$\mathbb{Q}(\sqrt{2} + \sqrt{5}) \cong \mathbb{Q}[x]/(x^4 - 14x^2 + 9)$

**Pf:** As $\dim_K K = n$, the elt $1, \alpha, \alpha^2, \ldots, \alpha^n$ must be linearly dependent, i.e. $\exists a_i \in F$ with

$$a_0 \cdot 1 + a_1 \alpha + \cdots + a_n \alpha^n = 0$$

Set $p(x) = a_0 + a_1 x + \cdots + a_n x^n \in F[x]$. If $p(x)$ is reducible, replace it by an irreducible factor for which $\alpha$ is also a root. □
Note: A posteriori, \( p \) must be non-nil. as
\[
\left[ \frac{F[x]}{(g(x))} : F \right] = \deg g.
\]

Ex: \( \mathbb{Q}(\sqrt{2}, \sqrt{5}) \) has \( \mathbb{Q} \)-basis \( 1, \sqrt{2}, \sqrt{5}, \sqrt{10} \).

We compute
\[
\begin{align*}
\alpha &= \sqrt{2} + \sqrt{5} \\
\alpha^2 &= 7 + 2\sqrt{10} \\
\alpha^3 &= 17\sqrt{2} + 11\sqrt{5} \\
\alpha^4 &= 89 + 28\sqrt{10}
\end{align*}
\]

\[\Rightarrow \alpha^4 - 14\alpha^2 + 9 = 0\]

What about simple extensions where \( [F(\alpha) : F] = \infty \)?

Then \( p(\alpha) \neq 0 \) for all non-zero \( p \in F[x] \).

Ex: \( \mathbb{Q}(\pi) \), \( \mathbb{Q}(e) \).

Ex: Field of rational fns with base field \( F \).

\[
F(x) = \text{fraction field} = \left\{ \frac{p(x)}{q(x)} \mid p, q \in F[x], q \neq 0 \right\}
\]

\[ [F(x) : F] = \infty \text{ since } 1, x, x^2, \ldots \text{ are linearly indip} / F. \]
Any simple ext $F(\alpha)/F$ of co degree is isomorphic to $F(\alpha)$, because

$$\phi: F(x) \rightarrow F(\alpha)$$

$$\frac{p(x)}{q(x)} \rightarrow \frac{p(\alpha)}{q(\alpha)}$$

makes sense as $q(\alpha) \neq 0$ if $q \neq 0$ in $F[x]$. As before, $\phi$ must be an isomorphism. So $F(\alpha) \cong F(x)$.

Cor. $Q(\pi), Q(e), Q(ln 2)$ are all isom. fields (to $Q(x)$).