Lecture 39:

Ex from last time: \( V(y - x^2) = V \leq C^2 \)

Consider \( h: V \rightarrow \{y\text{-axis}\} \), i.e. \( h(x, y) = y \)

in \( C[V] \). Given a ring homomorphism \( h^*: C[t] \rightarrow C[V] \)

by \( h^*(f(t)) = f(h(x, y)) = f(y) \). So \( h^*(t) = y \).

Get a 1-1 field homomorphism \( h^*: C(t) \rightarrow C(V) \)

Set \( F = h^*(C(t)) = C(y) \) and \( K = C(V) \).

The field extension \( K/F \) is

1. Simple as \( K = F(x) \).
2. Algebraic as \( x \) is a root of \( z^2 - y \in F[z] \).
3. \( z^2 - y \) is irreducible in \( F[z] \) by Eisenstein with \( R = C[y], I = (y) \).

So
\[
K = \frac{F[z]}{(z^2 - y)} = F(\sqrt{y}).
\]

Fun Fact: As abstract fields, \( K \cong F \). Specifically, if we project onto the \( x \)-axis instead by
\[ g(x, y) = x, \text{ we get } C[t] \rightarrow C[V] \text{ which } \]
\[ t \rightarrow x \]
is onto since \( y = x^2 \) in \( C[V] \). Thus we get an \underline{isomorphism} of fields
\[ C(t) \rightarrow C(V). \]

Not as weird as it seems: Note \( C(t^2) \subseteq C(t) \)
but \( C(t) \rightarrow C(t^2) \) is an isom.

Same reasoning shows in general:

\textbf{Thm}: \( V = V(f) \subseteq C^2 \) an \underline{irreducible} plane curve.
Then \( C(V) \) is a finite extension of \( C(t) \).

This has a partial converse:

\textbf{Thm}: Suppose \( K \) is a finite extension of \( C(t) \).
Then \( \exists \) an \underline{irreducible smooth curve} \( V \subseteq C^n \)
where \( C(V) = K \).

[Such fields are called \underline{function fields}.]
Back to the example: $K/F$ is Galois with group $G = \mathbb{Z}/2\mathbb{Z}$ whose gen sends $x = \sqrt{y} \mapsto -x = -\sqrt{y}$.

This corresponds to a symmetry of $V$:

Goal: Given a finite group $G$, build $K/C(t)$ with Galois group $G$.

Outline: [Reverse the above.]

1. Given $G$, find a curve $V$ (in $\mathbb{C}^n$ or $\mathbb{P}_\mathbb{C}^n$) where $G$ acts as a group of symmetries of $V$.

2. Each $\sigma \in G$ gives an automorphism of $C(V)$. [Think of $C(V)$ as functions on $V$]

3. Identify $C(V)_G$ with $C(V/\sigma)$ where $V/\sigma$ is the quotient, which is an alg. curve.

4. Do 1 so that $V/\sigma = \mathbb{P}_\mathbb{C}$ and hence $C(V/\sigma) = C(t)$. Thus we have an extension $C(V)/C(t)$ with Galois group $G$. 
Thinking about \( \mathbb{C} \):

\[
\mathbb{C} = \mathbb{R}^2 \times \mathbb{R}^2
\]

Back to example:

Symmetry: \( x \rightarrow -x \)

\[
V \xrightarrow{h} \mathbb{C} \\
(x, y) \rightarrow y
\]

If we identify \( V \) with \( \mathbb{C} \) by projection onto the \( x \)-axis, the map \( h \) becomes \( \mathbb{C} \rightarrow \mathbb{C} \)

\[
z \mapsto z^2 \text{ for } z \in \mathbb{C} \]

\[
\infty \mapsto \infty
\]

Let \( \overline{V} = \mathbb{P}^2_C \) be the con. proj. curve. Have \( \overline{V} = \mathbb{P}^1_C \). We want to consider the conic.

map \( \overline{h}: \overline{V} = \mathbb{P}^1_C \rightarrow \mathbb{P}^1_C \)

\[
z \mapsto z^2 \text{ for } z \in \mathbb{C} \]

\[
\infty \mapsto \infty
\]
This is a polynomial map since $\mathbb{P}^1_\mathbb{C} \to \mathbb{P}^1_\mathbb{C}$

$$(u:v) \mapsto (u^2:v^2)$$

restricts to $\mathbb{C}^2$ as $u \mapsto u^2$ and sends $\overline{(1:0)} \to (1:0)$. What does it look like? First note $\overline{h}$ sends

$(u:v)$ and $(-u:v)$ to the same pt. In pictures

Map on the equator looks like

$z \mapsto -z$

rotate by $\pi$

Compare:

$\mathbb{C} \to \mathbb{C}$

$z \mapsto z^2$

$z = re^{i\theta} \mapsto r^2e^{i2\theta}$
This is like a cone, but there is too much angle around the cone pt.

On $\mathbb{P}_C^1$, have

It is an example of a braneheel cover: a map that's locally 1-1 except at a few points where it looks like $\mathbb{Z} \to \mathbb{Z}$. 