Lecture 31: Varieties and ideals

$k$ = field
Affine space: $k^n$
$I \subseteq k[x_1, ..., x_n]$

Algebraic Variety: $\mathbb{V}(I) = \{ a \in k^n | f(a) = 0 \ \forall f \in I \}$

I might as well be an ideal.

Basic Props
1. $I \subseteq J \Rightarrow \mathbb{V}(I) \supseteq \mathbb{V}(J)$
2. $\mathbb{V}(I) \cap \mathbb{V}(J) = \mathbb{V}(I \cup J) = \mathbb{V}(I + J)$

Ex: $k = \mathbb{R}$, $n = 2$, $I = (x-y)$, $J = (x+y)$
$I + J = (x, y)$

3. $\mathbb{V}(I) \cup \mathbb{V}(J) = \mathbb{V}(I \cdot J)$

Ex: $I \cdot J = ((x-y)(x+y))$
$= \{ f(x,y)(x-y)(x+y) \}$
Let $V$ be an algebraic variety. Set

$$
\mathbb{V}(V) = \{ f \in k[x_1, \ldots, x_n] \mid f(a) = 0 \text{ for all } a \in V \}
$$

**Note:** $\mathbb{V}(\mathbb{V}(V)) = V$ since if $V = \mathbb{V}(I)$ then $\mathbb{V}(V) \supseteq I$ and every $f \in \mathbb{V}(V)$ vanishes on $V$.

**Key point:** $\mathbb{V}(\mathbb{V}(I)) \supseteq I$ but not always equal!

**Ex:** $I = (x^2) \subseteq k[x] \overset{0}{\longrightarrow} k$

$\mathbb{V}(I) = \{0\}$ but $\mathbb{V}(\{0\}) = (x)$.

*In practice, this is a real problem...*

**Def:** For $I$ an ideal in a (commutative) ring $R$, its **radical** is $\text{rad}(I) = \{ a \in R \mid a^n \in I \}$

**Ex:** $\text{rad}( (x^2) ) = (x)$ "Zero locus thin."

---

**Hilbert's Nullstellensatz:** Suppose $k$ is alg. closed.

Then $\mathbb{V}(\mathbb{V}(I)) = \text{rad}(I)$ for all ideals $I \subseteq k[x_1, \ldots, x_n]$. Moreover we have inverse bijections

$$
\begin{align*}
\{ \text{Algebraic varieties in } k^n \} & \overset{\mathbb{V}}{\longrightarrow} \{ \text{radical ideals in } k[x_1, \ldots, x_n] \} \\
\{ \text{in } k[x_1, \ldots, x_n] \} & \overset{\mathbb{V}}{\longleftarrow} \{ \text{Algebraic varieties in } k^n \}
\end{align*}
$$
Easy half of Pf: \( \Pi(V(I)) \supseteq \text{rad}(I) \)

Suppose \( f \in \text{rad}(I) \), i.e. \( f^n \in I \). If \( a \in V(I) \), then \( 0 = f^n(a) = (f(a))^n \Rightarrow f(a) = 0 \).

So \( f \in \Pi(V(I)) \).

Other half: next lecture.

\[\text{Ex: } I = (x^2 - 2) \subseteq \mathbb{Q}[x]. \text{ Then } \Pi(V(I)) = \mathbb{Q}[x] \text{ since } V(I) = \emptyset.\]

\[\text{Ex: } I = (x^2 + 1) \subseteq \mathbb{R}[x] \]

\[\Pi(V(I)) = \mathbb{R}[x].\]

\[\text{Nullstellensatz II: } k \leq \overline{k} \text{ with } \overline{k} \text{ algebraically closed. If } I \subseteq k[x_1, \ldots, x_n], \text{ then }\]

\[\Pi_k(V_k(I)) = \text{rad}(I).\]
Decomposing varieties:

\[ I = (x^3 + xy^2 - yx^2 - y^3 - x + y) \subseteq \mathbb{R}[x, y] \]

Turns out \( V(I) = V(x - y) \cup V(x^2 + y^2 - 1) \)
and in fact \( I = (x - y)(x^2 + y^2 - 1) \).

Can \( V(x - y) \) also be written as a union of two varieties?

**Def:** A variety \( V \) is **irreducible** if whenever \( V = V_1 \cup V_2 \) for varieties \( V_i \), then \( V = V_1 \) or \( V = V_2 \).

**Thm:** \( V \) is irreducible iff \( \mathbb{I}(V) \) is prime.

**Proof:** \( (\Rightarrow) \) Suppose \( f_1, f_2 \in \mathbb{I}(V) \). Set

\[ V_i = V \cap V(f_i) = V(\mathbb{I}(V) + (f_i)) \]

\[ = \{ \text{points of } V \text{ where } f_i = 0 \} \]

For \( a \in V \), we have \( (f_1 \cdot f_2)(a) = f_1(a) \cdot f_2(a) = 0 \)

\[ \Rightarrow f_1(a) = 0 \text{ or } f_2(a) = 0. \text{ So } V = V_1 \cup V_2. \]
As $V$ is irreducible, must have one $V_i = V$, say $V_1$. Thus $f_i(a) = 0$ for all $a \in V \Rightarrow f_i \in \Pi(V)$. So

$\Pi(V)$ is prime.

($\Leftarrow$) Suppose $V = V_1 \cup V_2$. Assume $V \neq V_1$. As $V_1 \not\subseteq V$, have $\Pi(V_1) \neq \Pi(V)$.

[Apply $V$ and use $V(\Pi(V)) = V$]

Pick $f_i \in \Pi(V_1) \setminus \Pi(V)$.

Suppose $f_2 \in \Pi(V_2)$. Then

$f_1 f_2 = 0$ on $V \Rightarrow f_1 f_2 \in \Pi(V)$. V

As $\Pi(V)$ is prime, must have one $f_i \in \Pi(V)$ which must be $f_2$. Hence $\Pi(V_2) \subseteq \Pi(V)$ and so $V_2 \supseteq I \Rightarrow V = V_2$. So $V$ is irreducible. \qed