Lecture 15: First proof of Poincaré Duality

Let $M$ be an $n$-manifold with simplicial triangulation $\mathcal{T}$ (so $\mathcal{T} = \Delta^n$'s with $n-1$ faces glued in pairs). For each $\sigma$ in $\mathcal{T}$,

$$\sigma \in \mathcal{T} \quad \Rightarrow \quad \begin{cases} D(\sigma) = \bigcup \{ \text{int}(\alpha) \mid \alpha \text{ in } sd(\sigma) \text{ has last vertex } \sigma \} \\ \overline{D}(\sigma) = \bigcup \{ \alpha \mid \text{same} \} \\ \mathring{D}(\sigma) = \overline{D}(\sigma) - D(\sigma) \end{cases}$$

[Draw 2-d and 3-d pictures]

- $\sigma \in \mathcal{T}$ are disjoint with union $M$.
- $D(\sigma)$ is a subcomplex of $sd(\mathcal{T})$ of dim $n-1$.
- $D(\sigma) = \{ D(\tau) \mid \sigma \neq \tau \}$, $D$ = "cell" complex of the $D(\sigma)$.

Def: $\mathcal{T}$ is PL if every $D(\sigma)$ is homeo to a ball $D^{n-1}$.

[I will assume this but it's not actually needed; independent of the topology of the $D(\sigma)$, homologically they are balls; also any smooth manifold has one...]

Thm: $M$ is a connected $n$-manifold with a PL triangulation $\mathcal{T}$. Then

$$H^k(M; \mathbb{F}_2) \cong H_{n-k}(M; \mathbb{F}_2)$$
Proof:

Can think of $D_{n-k}$ as a union of $n-k$ cells of $\mathcal{D}$ (as coeffs in $\mathbb{F}_2$).
Same with $C^k$ as union of $k$ cells of $\mathcal{J}$. So for $\sigma_k$ in $\mathcal{J}$ have both $\sigma_k^* \in C^k$ and $D(\sigma_k)$ in $D_{n-k}$.
and hence horizontal isomorphisms. It remains to show
that the diagram commutes. First,

$$S\sigma^* = \bigcup \{ \tau \mid |\tau| = |\sigma| + 1 \}$$

Second, topologically the boundary of $D(\sigma)$
is $D(\sigma) = \bigcup \{ D(\tau) \mid \tau \supset \sigma \}$. Hence

$$\partial D(\sigma) = \bigcup \{ D(\tau) \mid \tau \supset \sigma \text{ and } |\tau| = |\sigma| + 1 \}$$

So the diagram commutes!
Poincaré Duality holds!
Same proof ("turn cell decomposition upside down") works for \( \mathbb{Z} \), if we orient things carefully. [There's an annoying inductive way to do this...]

\[ \text{Cap product in homology: Continue with } \mathbb{F}_2 \text{ coeffs.} \]

\[ H_k(M) \times H_{n-k}(M) \rightarrow \mathbb{F}_2 \]

\[ \alpha \quad \beta \quad \alpha \cap \beta \]

where \( c \in C_k \) and \( d \in D_{n-k} \) we have

\[ \alpha \cap \beta = \#(c \cap d) \mod 2. \]

[This well-defined and makes sense over \( \mathbb{Z}/\mathbb{R} \)]

Thm: \( M \) closed connected with PL triangulation \( J \). Then \( \cap \) is a non-degenerate bilinear form on \( H_k(M) \times H_{n-k}(M) \).

Pf: Pick \([\varphi] \) in \( H^k(M) \) with \( \varphi \in C^k \) and \( \varphi(c) = 1 \). Then

\[ c \cap D(\varphi) = \varphi(c) = 1. \]
Also, if $\eta$ and $\psi$ are the Poincaré duals of $\alpha$ and $\beta$, then you can check that

$$\alpha \wedge \beta = (\eta \cup \psi)[M] \quad (\Rightarrow \text{Invariance of } \wedge \text{ on homology})$$

[For general $M$, once have $P.D.$ can use to define cap prod.]

Now easy to see that $H^*(\mathbb{R}P^n; F_2) = F_2[\alpha] / |\alpha| = 1$.

Types of manifolds:

- $\text{TOP:}$ Topological manifolds and cont maps.
- $\text{PL:}$ Mflds with PL-triangulations and PL-maps.
- $\text{DIFF:}$ Smooth mflds and maps.

In general, neither are injective or surjective...

Next: How to prove Poincaré Duality inductively.