Lecture 14: Poincaré Duality via triangulations.

**Cap Product:** \( H_k(X) \times H^l(X) \to H_{k-l}(X) \) for \( l \leq k \).

\( \sigma : \Delta^k \to X \quad \phi \in C^l(X) \)

\( \sigma \wedge \phi = \phi(\sigma |_{[v_0, \ldots, v_k]}) \sigma_{[v_k, \ldots, v_n]} \)

**Poincaré Duality:** \( M \) is \( R \)-orientable, \([M] \in H^n(M)\) a generator. Then \( D : H^k(M) \to H_{n-k}(M) \)

\( \phi \mapsto [M] \wedge \phi \)

is an isomorphism.

[Will give two proofs... starting with one that Poincaré would recognize...]

Let \( M \) be a closed \( n \)-manifold with a triangulation \( T \).

**Dual cell decomposition** \( \hat{T} \)

- **k-simplex** in \( T \)
- **n-k cell** in \( \hat{T} \)

Reverses inclusion relations

\( \sigma_0 < \sigma_1 \Rightarrow D(\sigma_0) > D(\sigma_1) \)

is a sub-simplex/cell
In the end, we'll get an isomorphism
\[
\begin{array}{ccc}
\text{cohomology complex of} & \Rightarrow & \text{homology complex w.r.t. } D_*
\end{array}
\]

of chain complexes, proving Poincaré Duality.

Consider a simplex
\[
\Delta = [v_0, v_1, \ldots, v_n] \subseteq \mathbb{R}^m
\]
The barycenter of \( \Delta \) is \( \hat{\Delta} = \frac{1}{n+1} \sum v_i \)
The barycentric subdivision \( sd(\Delta) \) of \( \Delta \) is
\[
\Delta = \bigcup \left\{ [\hat{\Delta}, w_0, \ldots, w_{n-1}] \mid \text{is a simplex in } sd(\partial \Delta) \right\}
\]
\[
sd(\Delta) =
\]
Can also do this to a \( \Delta \)-complex!
A $\Delta$-complex $X$ is simplicial if any subset $\{v_0, \ldots, v_n\}$ of $X^{(0)}$ is the vertices of at most one $n$-cell in $X$: 

Yes: 

No: 

[Go back to torus exp.]

Lemma: Any $\Delta$ complex can be made simplicial by subdividing twice.

[A simplicial $\Delta$-complex is more usually called] 

a simplicial complex.

Let $T$ be a simplicial complex structure on $M^n$ consisting of $\Delta^n$'s glued along faces. Each simplex $\sigma$ in $sd(T)$ has vertices which we order $[\hat{\alpha}_{i_1}, \hat{\alpha}_{i_2}, \ldots, \hat{\alpha}_{i_k}]$. Here, the "last vertex" $\hat{\alpha}_{i_1} > \hat{\alpha}_{i_2} > \ldots > \hat{\alpha}_{i_k}$ is the barycenter of the lowest dimension simplex involved.

For $\sigma$ in $T$ define

$D(\sigma) = \bigcup \{ \text{int}(\alpha) \mid \alpha \leq sd(T) \text{ with } \sigma \text{ as the last vertex} \}$
\( \tilde{D}(6) = \text{closure of } D(6) \)
\[= \bigcup \{ x \in \text{sd}(J) \mid \text{last vertex} \in \hat{6} \} \]
\( \hat{D}(6) = \tilde{D}(6) - D(6) \)

**Lemma:** @ The \( D(6) \) are disjoint and their union is \( M \)

\( \circ \) \( \tilde{D}(6) \) is a subcomplex of \( \text{sd}(J) \) of dimension \( n - k \) where \( |6| = k \).

\( \circ \) \( \hat{D}(6) = \{ D(\tau) \mid \tau \geq 6 \} \)

**Pf:** @ Every \( \alpha \) in \( \text{sd}(J) \) has a unique last vertex.

\( \circ \) If \( |6| = k \), then in some \( \Delta^n \) of \( \tilde{J} \)

and \( \alpha \in \tilde{D}(6) \) can have at most \( n - k + 1 \) vertices and hence dimension \( \leq n - k \).

\( \circ \) If \( \alpha \in \tilde{D}(6) - D(6) \), let \( \beta \in \text{sd}(J) \) have \( \hat{6} \) as the last vertex and \( \alpha < \beta \). Since \( \alpha \notin D(6) \), \( \alpha \) has last vertex \( \hat{\tau} \) for some \( \tau \in \tilde{J} \) distinct from \( 6 \). From \( \circ \) we get that \( \tau > 6 \), as needed.
Part of $D([v_0, v_1, i])$.

$D([v_0, ..., v_3])$

$D([v_0]) - v_1$

Note that $\overline{D}(s) = \text{Cone over } D(s)$. 