Lecture 11: Orientations and $H_n(M; \mathbb{Z})$

$H_n(X \cup A) = H_n(X, X \setminus A)$ (local homology)

$M$ an $n$-mfd. A local orient of $M$ at $x$ is a gen $\omega_x \in H_n(M \setminus \{x\}; \mathbb{Z}) = \mathbb{Z}$. An orientation of $M$ is a fn $x \mapsto \omega_x$ s.t. $\forall$ bounded $B \subseteq \mathbb{R}^n \subseteq U \subseteq M$ there is a $\sigma \in H_n(X \cup B)$ so that $\forall x \in B$ one has:

$$H_n(M \setminus B) \xrightarrow{i} H_n(M \setminus \{x\}) \xrightarrow{\nu} H_n(M \setminus \{x\})$$

[Also put up this on page 3 as a goal.]

Generator $[\sigma]$ of $H_n(M \setminus \{x\}) = H_n(M, M \setminus \{x\})$

Motivation: $\mathbb{RP}^2$ nonorientable, $S^2$ orientable

HW due next wed:
Define for any \( n \)-mfd \( \tilde{M} \)
\[
\tilde{M} = \{ u_\mathcal{B} \mid \mathcal{B} \subseteq M, u_\mathcal{B} \text{ a local chart at } \mathcal{B} \}\text{ topologized via } \forall \mathcal{B} \text{ ball } \Rightarrow \mathbb{R}_\text{open} \subseteq M \text{ and given } m_\mathcal{B} \in H_n(M|\mathcal{B}) \text{ we declare the following to be open}
\]
\[
U(m_\mathcal{B}) = \{ u_\mathcal{B} \mid \mathcal{B} \subseteq B \text{ and } m_\mathcal{B} \mapsto u_\mathcal{B} \}
\]
\[
H_n(M|\mathcal{B}) \mapsto H_n(M|\mathcal{B})
\]
\[
\tilde{M} \xrightarrow{p} M, \text{ which } u_\mathcal{B} \mapsto x
\]

it is easy to check is a covering map. \([Q: \text{Degree } = 2]\]

Prop: \( \tilde{M} \) is orientable, via taking as the orientation \( \tilde{u}_x \in H_n(\tilde{M}|u_x) \cong H_n(U(B)|u_x) \xrightarrow{p_*} H_n(B|x) \)

Prop: Suppose \( M \) is connected. Then \( M \) is orientable iff \( \tilde{M} \) has connected components.

Cor: If \( \pi_1 M = 1 \), then \( M \) is orientable. \([S^n, CP^n]\)
Define with analogous topology the larger covering space
\[ \tilde{M}_Z = \{ \alpha_x \in H_n(M|_x) \mid x \in M \} \cong \tilde{M} \]

[Can define for any ring.]

For orient \( M \), this is \( M \times \mathbb{Z} \). A \text{section} of
\[ M_Z \rightarrow M \] is a \text{cont map} \( s : M \rightarrow M_Z \) where \( p \circ s = \text{id}_M \).

Ex: \( s : x \mapsto 0 \in H_n(M|_x) \).

Thm. \( M^n \) closed connected. Then \( \mathbb{Q}_k(M; \mathbb{Z}) = 0 \ \forall k > n \).

6) If \( M \) is orientable, then \( H_n(M; \mathbb{Z}) \rightarrow H(M|_x; \mathbb{Z}) \)

is an isom \( \forall x \in M \). In particular, \( H_n(M; \mathbb{Z}) = \mathbb{Z} \).

6) Otherwise \( H_n(M; \mathbb{Z}) = 0 \).

Lemma: \( A^+ \subseteq M^n \). Then

1) If \( x \mapsto \alpha_x \) is a section of \( M_Z \rightarrow M \), then

\( \exists! \alpha_A \in H_n(M|A) \) whose image in \( H_n(M|_x) \) is \( \alpha_x \)

for all \( x \in A \).

2) \( H_k(M|A) = 0 \) for all \( k > n \).
Lemma $\Rightarrow$ Thm: Part 4 follows from 2 with $A = M$.  

Let $\Gamma(M)$ be the set of sections of $M_Z \to M$, which is a $\mathbb{Z}$-module. Have a homomorphism

$$H_n(M) \to \Gamma(M) \text{ by } \alpha \mapsto (x \mapsto i_\alpha \in H_n(M|x))$$

By 1, this is an isomorphism. Since $M$ is connected, a section is determined by its value at some fixed $p \in M$. When $M$ is orient, $M_Z = M \times \mathbb{Z}$ so $H_n(M) = \mathbb{Z}$ and each $H_n(M) \to H_n(M|x)$ is an isom.

If $M$ is non-orient, the only section is the zero section. So $H_n(M) = 0$.

Outline of Pf of Lemma:

1 [Key] True for $A, B, A \cap B \Rightarrow$ true for $A \cup B$.
2 Suffices to consider $M = \mathbb{R}^n$.
3 Holds for convex $A \subseteq \mathbb{R}^n$, hence unions of such.
4 Step 3 $\Rightarrow$ holds for all $cpt \subseteq \mathbb{R}^n$.
Note that (1) implies that if true for $A_i^{c^\text{pt}} \subseteq \bigcap_{i=1}^{m} \bar{A}_i$, and all $A_{i_1} \cap \ldots \cap A_{i_k}$ true for $\bigcap_{i=1}^{m} \bar{A}_i$.

For (2), by compactness any $A \subseteq M$ is a finite union $\bigcup_{i=1}^{m} A_i$ where each $A_i \subseteq U_i$ open $\subseteq \mathbb{R}^n$.

By excision, $H_n(M \setminus A_i) \cong H_n(U_i \setminus A_i)$ and also for any intersection $A_i \cap \text{ others }$. So if true for $c^\text{pt} A \subseteq \mathbb{R}^n$ done.

(3) If $A \subseteq \mathbb{R}^n$ is convex, then pick $x \in A$. Both $\mathbb{R}^n - A$ and $\mathbb{R}^n - \{x\}$ def. retract to a large sphere centered at $x$.

Hence $H_\ast(\mathbb{R}^n \setminus A) \rightarrow H_\ast(\mathbb{R}^n \setminus \{x\})$ is an isom.

[Query: Where did I use convexity?]