Lecture 10: Homology of Manifolds

Def: An n-manifold is a Hausdorff, 2nd countable, topological space where every pt has an open nbhd homeo to $\mathbb{R}^n$.

[Geometric topology: study of such. For now, $H_*$ and $H^*$]

$\checkmark = cpt$ and w/o boundary

Poincaré Duality: $M$ a closed, connected $n$-mfd.

Then $H_k(M; \mathbb{F}_2) \cong H_{n-k}(M; \mathbb{F}_2)$. If $M$ is orientable then $H_k(M; \mathbb{Z}) \cong H^{n-k}(M; \mathbb{Z})$.

[Surprising since being a mfd is a purely local cond.]

Thm. $M$ closed conn $n$-mfd. Then $H_n(M; \mathbb{Z}) = \mathbb{Z}$ or 0 and $H_n(M; \mathbb{F}_2) = \mathbb{F}_2$.

Def. A triangulation of $M$ is a $\Delta$-complex str consisting of $n$-simplices w/ their $n$-1 faces glued in pairs.

Ex: $n = 1$ -- -- -- -- $\to \triangle$
Suppose $M$ has a triangulation. Then $H_n(M; \mathbb{F}_2) = \mathbb{F}_2$.


What about $H_n(M; \mathbb{Z})$?

Either everything fits together: $H_n(M; \mathbb{Z}) = \mathbb{Z}$

or not

$H_n(M; \mathbb{Z}) = 0$

Moebius band
Q: Does every cpt n-mfld have a triangulation?  
A: No! [Manolescu 2013] [Every smooth one does, though.

Orientation of $\mathbb{R}^n$: [preserved under rotations, switch under reflections]

An orient of $\mathbb{R}^n$ at $x$ is a choice of generator in

$$H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}; \mathbb{Z}) \cong H_{n-1}(\mathbb{R}^n - \{x\}) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}$$

long exact seq.

Suppose $B$ open ball with $x \in B$.

$$H_n(\mathbb{R}^n, \mathbb{R}^n - \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n - B)$$

Local homology of $X$ at $A$:

$$H_n(X \setminus A) = H_n(X, X \setminus A)$$

If $M$ is an n-mfld

$$H_n(M \setminus x) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$$

$\Rightarrow [Q: Excision]

A local orient at $x \in M$ is a choice of gen $M_x$ of $H_n(M \setminus x) \cong \mathbb{Z}$. 
Def: An orientation of \( M \) is a map \( x : U \rightarrow M \) such that for open sets \( U \subseteq \mathbb{R}^n \) and bounded open balls \( B \subset U \), there exists \( u \in H_n(M \setminus B) \) such that

\[
H_n(M \setminus x) \leftarrow H_n(M \setminus B) \cong H_n(U \setminus B) \cong \mathbb{Z}
\]

for all \( x \in B \).

If an orientation exists, \( M \) is called orientable.

Thm: \( M \) closed connected \( n \)-mfld. If \( M \) is orientable, then \( H_n(M; \mathbb{Z}) = \mathbb{Z} \) and \( H_n(M; \mathbb{Z}) \to H_n(M \setminus x; \mathbb{Z}) \) is an isomorphism for all \( x \in M \). Otherwise,

\( H_n(M; \mathbb{Z}) = 0 \).

Note: 1) Easy to see that \( \mathbb{Z} \) implies orientability.

Fix a gen \( u \) of \( H_n(M; \mathbb{Z}) \) and set \( M_x = \text{image in } H_n(M \setminus x; \mathbb{Z}) \).

2) Any mfld is \( \mathbb{F}_2 \)-orientable, since

\[
H_n(M \setminus x; \mathbb{F}_2) \cong \mathbb{F}_2 \text{ has a single non-zero element.}
\]