Lecture 9: Applications of cohomology

Last time: $A, B$ are abelian gps
\[
A \otimes B = \bigoplus Z[a \otimes b]/(a + a') \otimes b = a \otimes b + a' \otimes b \quad a \otimes (b + b') = a \otimes b + a \otimes b'
\]

Division Algebras: Bilinear $R^n \times R^n \rightarrow R^n$ with no zero divisors.

\[
H^*(RP^n; \mathbb{F}_2) = \mathbb{F}_2[\gamma]/(\gamma^{n+1}) \quad |\gamma| = 1
\]

\[
H^*(RP^n \times RP^n; \mathbb{F}_2) = \mathbb{F}_2[\alpha, \beta]/(\alpha^{n+1}, \beta^{n+1})
\]

\[
\alpha = p_1^*(\gamma) \quad \beta = p_2^*(\gamma)
\]

Correction: $R$ ring, $A, B$ are $R$-modules

\[
A \otimes_R B = \bigoplus R[a \otimes b]/(a, b) \quad \text{same}
\]

Also an $R$-module because of
\[
+(ra) \otimes b = a \otimes (rb)
\]

Any abelian gp is a $Z$-module
\[
r \cdot a = \sum_{i=1}^r a
\]

For ab gps $A$ and $B$:

\[
A \otimes B = A \otimes ZB
\]

of groups as $Z$-modules.
Ex: $R \otimes_R R \cong R$ since $a \otimes b = a \otimes 1$

So $Q \otimes_Q Q \cong Q$ but $Q \otimes_Z Q$ is much larger.

$C \otimes_C C = C$ but $C \otimes_R C \cong R^4$.

Künneth Thm: Suppose $X$ and $Y$ are spaces where $H^*(Y; R)$ is a finitely generated free $R$-module.

Then $H^*(X \times Y; R) \cong H^*(X; R) \otimes H^*(Y; R)$.

Thm: If $R^d$ has the str of a division algebra, then $d = 2^k$.

Pf: Take $n = d - 1$.

Set $g: S^n \times S^n \to S^n$ to be $g(x, y) = \frac{x \cdot y}{|x \cdot y|}$.

[Makes sense because no 0-divisors.]

As $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$ we have $g(-x, y) = -g(x, y) = g(x, -y)$.

and so get a map $h: P^n \times P^n \to P^n$ $[P^n = \mathbb{R}P^n]$

Claim: With $\mathbb{F}_2$ coeffs $H'(P^n \times P^n) \xrightarrow{h^*} H'(P^n)$

$\alpha + \beta \xrightarrow{\gamma \times 1} \alpha \times \gamma$
Note because of the cup product, this completely determines \( H^*(\mathbb{P}^n \times \mathbb{P}^n) \leftarrow H^*(\mathbb{P}) \).

**Proof of Claim:** Take \( n > 1 \) so that \( \pi_* \mathbb{P}^n = \mathbb{Z}/2\mathbb{Z} \).

Let's compute \( \pi_* (\mathbb{P}^n \times \mathbb{P}^n) \xrightarrow{h_*} \pi_* (\mathbb{P}^n) \). Now

\[
\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \xrightarrow{\mathbb{Z}/2\mathbb{Z}}
\]

\( S^n \) gives a gen of \( \pi_* (\mathbb{P}^n) \).

What is \( h_* (1,0) \)? Well, \( (1,0) = (\lambda, \text{const } y_0) \)

\[
1 \xrightarrow{h_*} \lambda \cdot y_0 /
\]

Basically, changing \( \lambda \) by \( \lfloor \lambda \cdot y_0 \rfloor \)

the linear trans \(- \cdot y_0\), so still get a path joining antipodal pts and hence \( h_* (1,0) = 1 \) and \( h_* (0,1) = 1 \)

Same action on \( H_1 (\_, \mathbb{Z} \text{ or } \mathbb{F}_2) \) and since \( H'_* (\_, \mathbb{F}_2) = \text{Hom} (H_1 (\_, \mathbb{Z}), \mathbb{F}_2) \) we get the claim by dualizing.
Proof of Thm: As \( \gamma^d = 0 \) in \( H^*(P^n; \mathbb{F}_2) \) get

\[
0 = h^*(\gamma^d) = (\alpha + \beta)^d = \sum_{k=0}^{d} \binom{d}{k} \alpha^k \beta^{d-k}
\]

in \( H^*(P^n \times P^n; \mathbb{F}_2) \). So \( \binom{d}{k} \equiv 0 \mod 2 \) for all \( 0 < k < d \). Equivalently in \( \mathbb{F}_2[x] \) we have

\[
(1+\alpha)^d = 1 + \alpha^d
\]

Write \( d = d_1 + d_2 + \ldots + d_\ell \)

where each \( d_i \) is a power of 2 and \( d_1 < d_2 < \ldots < d_\ell \).

Then

\[
(1+\alpha)^d = (1+\alpha)^{d_1} \cdot \ldots \cdot (1+\alpha)^{d_\ell} = (1+\alpha^{d_1}) \cdot \ldots \cdot (1+\alpha^{d_\ell})
\]

since \( p_1 \rightarrow p_2 \) is an additive hom of \( \mathbb{F}_2[x] \)

= some poly with \( 2^\ell \) terms

\[\Rightarrow \ell = 1, \ i.e. \ d = 2^\ell. \]

If time remains, talk about Poincaré Duality:

**Def:** An \( n \)-manifold is a Hausdorff, 2\(^{nd}\) countable, topological space where every pt has an open mbhd homeo to \( \mathbb{R}^n \)

**Thm:** \( M \) compact connected \( n \)-mfdld. \( H_k(M; \mathbb{F}_2) \cong H_{n-k}(M; \mathbb{F}_2) \)

If \( M \) is orientable then \( H_k(M; \mathbb{Z}) \cong H_{n-k}(M; \mathbb{Z}) \).