Think about homological case: $n = 1$

\[ Y \xrightarrow{\beta} Y \times S^1 \]

**Pf. of Lemma:** Note that $H^n(Y) \xrightarrow{p^*} H^n(Y \times S^1) \xrightarrow{i} H^n(Y)$.

$p^*$ is 1-1 since the composition:

\[ \beta_1 \rightarrow \beta_1 \times 1 \]

\[ Y \xrightarrow{i} (Y \times pt^2) \subseteq Y \times S^1 \xrightarrow{p} Y \]

is the identity. Using the long exact seq for the pair $(Y \times S^1, Y \times \{pt\})$ shows that it is enough to understand

\[ Y \times S^1 \]

and prove that $H^n(Y) \rightarrow H^{n+1}(Y \times I, Y \times \partial I)$

\[ \beta \rightarrow \beta \times \alpha \]

is an isomorphism, where $\alpha \in H^1(I, \partial I) \cong H^1(S^1)$.
First:
\[ \Theta^I = \{ 0, 1 \} \quad \Rightarrow \quad H^0(\Theta^I) = \mathbb{R}^2 = \langle 1_0, 1_1 \rangle \]

\[ 0 \leftarrow H'(I, \Theta^I) \xrightarrow{\delta} H^0(\Theta^I) \leftarrow H^0(I) \leftarrow H^0(I, \Theta^I) \]

\[ 1_0 + 1_1 \leftarrow 1_1 \]

\[ \Theta(1_0) = (\text{unit in } \mathbb{R}) \times \]

Second: [Exercise] More generally, assume \( A \subseteq X \)
is reasonable, and consider

\[ (\delta \overline{\phi}, \psi) \]

\[ H^k(A) \oplus H^l(Y) \xrightarrow{\delta \oplus \text{id}} H^k(X) \oplus H^{l+1}(Y) \]

\[ \xrightarrow{x} \]

\[ H^{k+l}(A \times Y) \xrightarrow{\delta} H^{k+l+1}(A \times Y, X \times Y) \]

\[ \overline{\phi} \in C^k(X) \quad \text{and} \quad \psi \in C^l(A) \]

Reason: Start with cocycles \( \phi \in C^k(Y) \) and \( \psi \in C^l(A) \).

Extend \( \phi \) to \( \overline{\phi} \) in \( C^k(X) \). Conclude the equality \( \oplus \) since

\[ \delta(p^*_X(\overline{\phi}) \cup p^*_Y(\psi)) = (\delta p^*_X(\overline{\phi}) \cup p^*_Y(\psi)) + (-1)^k \phi \cup 5 \psi \]

\[ = 0. \]
$\beta \times (1, 1_x) = (\beta, \beta) \xrightarrow{\beta} H^n(Y_x, (y, 1), H^n(Y)) \xrightarrow{\beta} H^n(Y_x, (y, 1), H^n(Y)) \xrightarrow{\beta} H^n(Y_x, (y, 1), H^n(Y))$.

This map is just the diagonal map hence injective.

Note that $\beta \times 1_x : H^n(Y_x, 1, H^n(Y)) = \beta \times S(I_0) = (\beta \times 1_x)$ is an isomorphism.

By commutativity, we have $\beta \times 1_x : H^n(Y_x, 1, H^n(Y)) = \beta \times S(I_0) = (\beta \times 1_x)$.

This proves claim $\square$ and thus the lemma.
Lecture 7: Last time and first half of class: see "lecture 6" notes pgs 4-6.

HW#2: Due Wed Sept 24.
Hatcher:
and others to be assigned.

What is $H^*(X \times Y)$? Starting pt:

$H^*(X) \times H^*(Y) \xrightarrow{\times} H^*(X \times Y)$

$\alpha \quad \beta \quad \rightarrow \quad \alpha \times \beta = P_x^*(\alpha) \cup P_y^*(\beta)$

[Might hope this is an isomorphism, but...]

$X = S^1 \quad Y = ?$ pt? \quad $X \times Y = S^1$

$(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \oplus \mathbb{Z}_2 \rightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2$

Also, $\times$ is bilinear, not a homomorphism. That is

$(\alpha_1 + \alpha_2) \times \beta = \alpha_1 \times \beta + \alpha_2 \times \beta$ and reversed and so

$X((\alpha_1, \beta_1) + (\alpha_2, \beta_2)) = X((\alpha_1 + \alpha_2, \beta_1 + \beta_2))$

$= \alpha_1 \times \beta_1 + \alpha_1 \times \beta_2 + \alpha_2 \times \beta_1 + \alpha_2 \times \beta_2$

$\neq X((\alpha_1, \beta_1)) + X((\alpha_2, \beta_2))$