Poincaré Duality: $M^n$ an $R$-oriented mfld. Then

$$D_M: H^k_c(M; R) \to H_{n-k}(M; R)$$
is an $\cong$ for all $k$.

A Hausdorff 2nd countable topological space $M$ is a manifold with boundary if every $x \in M$ has an open nbhd $\cong$ to $R^n$ or $R^{n-1} \times [0, \infty)$.

$n=2$:

If $\partial M \neq \emptyset$, then $H_n(M) = 0$ since $M \cong_{h.e.} M \setminus \partial M$ and non-cpt mflds have $H_n = 0$.

**Thm:** $M^n$ an $R$-oriented mfld w/ bdry. Then

- $H^k(M) \cong H_{n-k}(M, \partial M)$
- $H^k(M, \partial M) \cong H_{n-k}(M)$
Geometric picture: [cup product on homology]

\[ H_1(S) \times H_1(S, \partial S) \to \mathbb{Z}, \text{ but not } H_1(S, \partial S) \times H_1(S, \partial S) \text{ since} \]

and so \(\cap\) product is not well defined. So

\[ H^1(S) = \text{Hom}(H_1(S)) \cong H_1(S, \partial S). \]

[Can prove this in either way, will use 2nd proof]

Prop: If \(M\) is a cpt mfld with bdry, then \(\partial M\) has an open nbhd \(\cong \to \partial M \times [0,1)\)

[Pf: See Hatcher.]

Cor: \(N = M \setminus (\partial M \times [0,1/2])\) is a def. retract of \(M\) and \(M - \partial M\). In particular, \(M = \text{n.e.} \frac{N}{\partial N}\)

Pf of thm: (8) Follows from \(H^k(M, \partial M) \cong H^k_c(M \setminus \partial M)\) via the Cor. In more detail, set
\[ M_n = M \setminus (\exists m \in [0, \frac{1}{n}]) \]

\[ H_*^k(M) = \lim_{n \to \infty} H_*^k(M \setminus M_n) \]

\[ H_*^k(M, M \setminus M_n) \cong H_*^k(M_n, \mathbb{Z}M_n) \]

Also, clearly \( H_{n-k}(M) = H_{n-k}(M, \mathbb{Z}M) \).

(A) follows from (B) via long exact sequences of the pair, which are comp. with \( [M, \mathbb{Z}] \).

Alexander Duality: If \( K \subseteq S^n \) is cpt, locally contractible and not \( \emptyset \) or \( S^n \), then for all \( i \):

\[ \tilde{H}_i(S^n \setminus K; \mathbb{Z}) \cong \tilde{H}_{n-i-1}(K; \mathbb{Z}) \]

Cor: \( \tilde{H}_*(S^n \setminus K) \) does not depend on how \( K \) is embedded in \( S^n \).

\[ S^1 \to S^3: \quad \circ \quad \bigcirc \quad \bigcirc \quad \text{or} \]
In each case, \( \tilde{H}^*(S^3, S') = \{ \mathbb{Z} \text{ if } i = 1 \} \) is finitely generated.

**Proof sketch:** Will show 

\[
H_i(S^n \setminus K) \cong H_c^{n-i}(S^n \setminus K) \quad \text{[Poincaré]}
\]

\[
\cong \lim_{U \supseteq \text{open}} H^{n-i}(S^n \setminus K | S^n \setminus U) \quad \text{[local co as lim]}
\]

\[
\cong \lim_{U} H^{n-i}(S^n, U) \quad \text{[Excision.]} \]

\( \mathbb{4} \) \( \cong \lim_{U} H^{n-i-1}(U) \) if \( i \neq 0 \)

\( \mathbb{5} \) \( \cong \tilde{H}^{n-i-1}(K) \)

Step \( \mathbb{4} \) is the long exact sequence of the pair \( (S^n, U) \):

\[
0 \hookrightarrow \tilde{H}^{n-i}(S^n) \rightarrow \tilde{H}^{n-i}(S^n, U) \rightarrow \tilde{H}^{n-i-1}(U) \rightarrow \tilde{H}^{n-i-1}(S^n) \rightarrow 0
\]

\( \Rightarrow \) using \( i = 0 \) here.
For part 5, need a little point-set topology.

If $K$ has an open subset $U$ which def retracts to $K$, then this is easy.

In general only have retraction but argument can be made to work.