Homotopy Lifting Property: \( p: E \to B \) has H.L.P with respect to \( X \) if given \( g_0: X \to B \) and a lift \( \tilde{g}_0 \) of \( g_0 \) to \( E \) such that \( p \circ \tilde{g}_0 = g_0 \), every homotopy \( \tilde{g}_t: X \times I \to B \) lifts to \( E \) starting at \( \tilde{g}_0 \).

Def: \( p: E \to B \) is a fibration if it has H.L.P with respect to all \( X \).

Ex: \( E = B \times F \), \( p \) projection onto \( B \). If \( \tilde{g}_0: X \to E \) is given by \( \tilde{g}_0(t) = (g_0(t), h(t)) \) then define \( \tilde{g}_t(x) = (g_t(x), h(t)) \).

[Query:] Ex: \( p: E \to B \) a covering space.

[In general, \( E \to B \) is a “twisted product” with fixed fiber \( F \).]
Thm: Suppose $p : E \to B$ is a fibration. Let $b_0 \in B_0$ and $x_0 \in F = p^{-1}(b_0)$. If $B$ is path connected the following is exact

$$
\cdots \to \pi_n(F, x_0) \overset{i_*}{\to} \pi_n(E, x_0) \overset{p_*}{\to} \pi_n(B, b_0) \\
\to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(F, x_0) \to \pi_0(E, x_0) \to 0.
$$

**Key claim:** $p_* : \pi_n(E, F, x_0) \to \pi_n(B, b_0, b_0) = \pi_n(B, b_0)$

is an isomorphism.

Combining with the long exact sequence of $(E, F)$, this gives the claim except for the very last 0; i.e. every path comp of $E$ contains a point of $F$.

Given $e \in E$, join $p(e)$ to $b_0$ by a path, a lift of this path to one starting at $e$ joins $e$ to some pt in $F$.

**Proof Sketch:** $p_*$ onto.
\[ f_\sim |_J = \text{const}_{\kappa_0} \]

Extend by H.L.P. to \( D^n \)

\[ f \in \pi_n(E,F,b_0) \]

\[ F = p^{-1}(b_0) \]

\[ p_*[f] = [\tilde{f}] \]

\[ p_\sim \]

\( P \) is 1-1: If \( p_\sim [\tilde{f}] = p_\sim [\tilde{g}] \) then use HLP to lift homotopy between \( p_\sim f \) and \( p_\sim g \) to see \( [\tilde{f}] = [\tilde{g}] \).

Fiber Bundles: [locally a product]

\[ E \rightarrow B \]

is a fiber bundle with fiber \( F \)

if each pt in \( B \) has a nbhd \( U \)

and a homeomorphism \( p^{-1}(U) \rightarrow U \times F \)

when the diagram \( \begin{array}{ccc} p^{-1}(U) & \xrightarrow{h} & U \times F \\ \downarrow \quad \circ \quad \downarrow \quad \text{proj into } U \\ U \end{array} \) commutes.

Fact: Fiber bundle maps are fibrations.
Ex: \( E = B \times F \)  

2) Covering space \((F = \text{discrete set})\)

Ex: Möbius band

\[ I \to M \to S^1 \]

Notation for fiber bundle

Ex: Hopf bundle: \( S^1 \to S^3 \to S^2 \)

\[ S^3 = \left\{ (z_0, z_1) \in \mathbb{C}^2 \mid |z_0|^2 + |z_1|^2 = 1 \right\} \]

\[ \mathbb{C}P^1 = \mathbb{C}^2 \setminus \{0\} / \mathbb{C}^* \]

\[ [z_0 : z_1] \]

For \( \lambda \in S^1 \subseteq \mathbb{C} \), note \( p(\lambda z_0, \lambda z_1) = p(z_0, z_1) \).

In fact \( p^{-1}(pt) = \text{circle} \), since if \((z_0, z_1) \in S^3 \) and \((\lambda z_0, \lambda z_1) \in S^3 \) then \(|\lambda| = 1\) by \( \star \)
Local triviality: Take $U \subseteq \mathbb{CP}^1$ to be $\{[z_0:1] \mid z_0 \in \mathbb{C} \} \cong \mathbb{C}$.

Define $h : \mathbb{P}^{-1}(U) \rightarrow U \times S^1$ by

$$h([z_0:z_1]) = ([z_0/|z_1|, z_1/|z_1|])$$

This is a homeo since here is the inverse:

$$h^{-1}(z_0:z_1) = \left(\frac{z_0}{\sqrt{1+|z_0|^2}}, \frac{z_1}{\sqrt{1+|z_0|^2}} \right)$$

So diagram commutes.

http://nilesjohnson.net/hopf.html