Lecture 28: Homology and homotopy.

Thm. For any $G$, there is a $K(G, 1)$. For abelian $G$, there is a $K(G, n)$ for all $n \geq 1$. These spaces are unique up to homotopy equivalence.

Lemma: $X = \bigvee_{\alpha} S^n_\alpha$ has $\pi_i = 0$ for $i < n$ and $\pi_n = \bigoplus_{\alpha} \mathbb{Z}$

Pf: If there finitely many $\alpha$, then $X$ is the $n$-skeleton of $\bigvee_{\alpha} S^n_\alpha$. Since all other cells of $\bigvee_{\alpha} S^n_\alpha$ have dim $\geq 2n$, get

$\pi_i X \cong \pi_i \bigvee_{\alpha} S^n_\alpha \cong \bigvee_{\alpha} \pi_i S^n_\alpha$ for $i \leq n$.

When $\omega$-many spheres $\bigvee_{\alpha} S^n_\alpha$ is not obviously a CW complex.

Consider $\bigoplus_{\alpha} \mathbb{Z} \xrightarrow{\Phi} \Pi_n X$ where $\mathbb{Z}_\alpha$ goes to the image of $\Pi_n S^n_\alpha \xrightarrow{i_{\alpha}} \Pi_n X$. $\Phi$ is onto since any $f: S^n \rightarrow X$ has image contained in a finite subwedge; same for homotopies, so $\Phi$ is 1-1.
For any space $X$ have homomorphisms

$$h: \pi_k X \to H_k(X; \mathbb{Z})$$

$$(f: S^k \to X) \mapsto f_*([S^k])$$

but often these aren't useful: $S^n$ has little $H_*$ much $\pi_*$

$CP^\infty$ lots of $H_*$, little $\pi_*$

Hurewicz Thm: If $Y$ is an $(n-1)$ connected CW complex where $n \geq 2$, then $\tilde{H}_i(Y) = 0$ for $i < n$ and $\pi_n Y \cong H_n(Y)$ via $h$.

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Pf: By Cor 4.16, can assume that $Y$ has a single 0 cell and no other cells of dim $< 2$.

Hence by cellular homology see that $\tilde{H}_i(Y) = 0$ for $i < n$. 
As both $H_n$ and $\pi_n$ are det by $Y^{(n+1)}$, can assume $Y$ has no cells of dim $>\ n+1$. So

$$Y = \left( \bigvee_{\alpha} S^n \right)_x \cup_{\beta} e^{n+1}_\beta$$

As with the $K(0,n)$ theorem, we have

$$\pi_{n+1}(Y,X) \xrightarrow{\partial} \pi_n(X) \rightarrow \pi_n(Y) \rightarrow \pi_n(Y,X)$$

Key: $\pi_{n+1}(Y,X)$ is gen by $D^{n+1} f^\beta \rightarrow e_{n+1}^\beta$

which has $\partial$ given by $S^n \xrightarrow{\phi^\beta} X$ where $\phi^\beta$ is the attaching map for $e_{n+1}^\beta$. 
The components of $\partial f_\beta = \phi_\beta \in \pi_n X = \bigoplus \alpha Z$
can be computed by the compositions

$S^n \xrightarrow{\phi_\beta} X \xrightarrow{P_\alpha} X / \bigvee_{\alpha \neq \alpha} S^n_\alpha = S^n_\alpha$

That is

$\partial f_\beta = \sum_{\alpha} [P_\alpha \circ \phi_\beta] \in \pi_n S^n_\alpha$

and $[P_\alpha \circ \phi_\beta] \in Z$ is just the degree of $P_\alpha \circ \phi_\beta$.

But this matches the cellular boundary formula

$\partial c^{n+1}_\beta = \sum_{\alpha} d_{\alpha \beta} \cdot S^n_\alpha$

degree of $c^{n+1}_\beta \to \bigvee_{\alpha} S^n_\alpha / \bigvee_{\alpha \neq \alpha} S^n_\alpha$

and so left hand square commutes.

Thus $\pi_n Y \xrightarrow{h} H_n Y$ is an isomorphism

by the 5-lemma. \[\square\]
Relative Hurewicz: \((X, A)\) a CW pair which is 
\((n-1)\) connected for \(n \geq 2\) with \(A\) simply 
connected and non-empty. Then 
\[H_i(X, A) = 0\] for \(i < n\) and \(\pi_n(X, A) \cong H_n(X/A)\).

\textbf{Pf:} By excision, \(\pi_i(X, A) \cong \pi_i(X/A)\) for 
\(i \leq (n-1) + 1 = n\). Of course \(H_i(X, A) \cong \widetilde{H}_i(X/A)\) 
for all \(i\). Now apply original Hurewicz to 
\(X/A\) which is \(n-1\) connected. \(\square\)
Cor: A map $f: X \to Y$ between simply connected CW complexes is a homotopy equivalence iff $f_*: H_n(X) \to H_n(Y)$ is an isomorphism for all $n$.

Pf: Replacing $Y$ by the mapping cylinder, we can assume $f$ is the inclusion of a subcomplex. Since $X$ and $Y$ are simply connected, we have that $\pi_1(Y, X) = 0$. The relative Hurewicz Theorem says that the first non-zero $\pi_n(Y, X)$ is isomorphic to the first non-zero $H_n(Y, X)$.

But all the $H_n(Y, X) = 0$. Hence all $\pi_n(Y, X) = 0$.

$\Rightarrow$ all $\pi_n X \to \pi_n Y$ are $\cong \Rightarrow X \cong_{h.e.} Y$ by Whitehead.