Lecture 27: Eilenberg-MacLane spaces

A $K(G,n)$ is a CW complex $X$ where $\pi_n X = G$ and all other $\pi_i X = 0$.

**Thm:** Any group $G$ has a $K(G,1)$. If $G$ is abelian, then $\exists K(G,n)$ for all $n$. These spaces are unique up to homotopy equivalence.

---

**Pf of existence:** $n = 1$.  

*Need to find $X$ with $\pi_1 = G$ and contractible univ. cover.*

$EG = \Delta$-complex with one $n$-simplex for each ordered $n+1$ tuple $(g_0, \ldots, g_n)$ of elts of $G$.  

*Vertices $\leftrightarrow G$ and simplices are glued by obv. restriction rel.*

**Ex:** $G = \mathbb{Z}/3\mathbb{Z}$

\[\text{Contractible: Given } x \in \Delta(g_0, \ldots, g_n) \text{ homotope to } 1_G \]

*Via the straight line in $\Delta(1, g_0, \ldots, g_n)$*  

*Doesn't matter which simplex we regard $x$ as in:*
$G$ acts freely on $EG$ by $\Delta(g_0, \ldots, g_n) \xrightarrow{h} \Delta(hg_0, hg_1, \ldots, hg_n)$ linear map.

Set $BG = EG/G$ a $\Delta$-complex. Then $EG \to BG$ is a covering map and $BG$ is a $K(G, 1)$.

\[
\begin{align*}
N > 1: \quad \text{Lemma:} \quad \bigvee_{\alpha} S^n \quad \text{has} \quad \pi_i &= 0 \quad i < n \quad [\text{Pf of this}] \\
\pi_n &= \bigoplus_{\alpha} \mathbb{Z} \quad [\text{below}].
\end{align*}
\]

Consider \(0 \to K \xrightarrow{i} \bigoplus_{\alpha} \mathbb{Z} \to G \to 0\) ident with $\bigvee_{\alpha} S^n$.

Attach, an $n+1$ cell to $\bigvee_{\alpha} S^n$ for each $\beta \in K$ via $\partial D^{n+1}_{\beta} \to \bigvee_{\alpha} S^n$ with $\varphi_{\beta} = i(\beta)$ to get a CW complex $X$.

\[
\begin{align*}
\pi_{n+1} (X, V_\alpha S^n_\alpha) &\xrightarrow{\partial} \pi_{n} (V_\alpha S^n_\alpha) \xrightarrow{} \pi_{n} (X) \xrightarrow{} \pi_{n} (X, V_\alpha S^n_\alpha) \\
\text{II} &\xrightarrow{\sim} \text{as } V_\alpha S^n_\alpha \text{ is } n-1 \text{ conn.} \\
\pi_{n+1} (X/V_\alpha S^n_\alpha) &\xrightarrow{} \pi_{n} (X, V_\alpha S^n_\alpha) \text{ is } n-\text{conn.} \quad \text{II} \\
\pi_{n+1} (V_\beta S^{n+1}_\beta) &\xrightarrow{} \pi_{n} (X) \xrightarrow{\sim} G \\
\bigoplus_{\beta} \mathbb{Z} &\xrightarrow{} \pi_{n} (X) \xrightarrow{\sim} G
\end{align*}
\]
So have $X$ which is $n-1$ connected and $\pi_n = 0$.

But $\pi_{n+1} X$ might be non-zero. If so, attach a bunch of $n+2$ cells along a gen set for $\pi_{n+1} X$. to get $X_2$. Note $\pi_i X_2 = \pi_i X$ for $i \leq n$ since $X^{(n+1)}_2 = X$. Lather, rinse, repeat, to get a $K(G, n)$.

Proof of Lemma: Suppose there are finitely many $\alpha$.

Then $\bigvee_{\alpha} S^n_\alpha \to \prod_{\alpha} S^n_{\alpha}$ as the $n$-skeleton, and all other cells of $\prod_{\alpha} S^n_{\alpha}$ have dim $\geq 2n$. So by cellular approx, $\pi_n (\bigvee_{\alpha} S^n_\alpha) \cong \pi_n (\prod_{\alpha} S^n_{\alpha}) = \bigoplus_{\alpha} \mathbb{Z}$.

[When there are infinitely many $\alpha$, the topological product $\prod_{\alpha} S^n_{\alpha}$ is not obviously a CW complex.]

Consider $\bigoplus_{\alpha} \mathbb{Z} \xrightarrow{\bigoplus_{\alpha} \pi_n} \pi_n \bigvee_{\alpha} S^n_\alpha$ where

$\mathbb{Z}_\alpha \to (\pi_n S_\alpha \to \pi_n \bigvee_{\alpha} S^n_\alpha)$
\( \Phi \) is surjective since any \( f : S^n \rightarrow V \wedge S^n \) has opt image hence contained in some finite wedge of \( S^n \).

\( \Phi \) is injective since any null homotopy is again contained in some finite wedge.

\[ \text{Q: What prevents us from doing the 2\textsuperscript{nd} construction?} \]
when \( n=1 \) ? Actually nothing!

\[ \text{Pf of uniqueness: Enough to show that any } K(6,n) \]
\( Y \) is homotopy equivalent to the one \( X \) constructed above. Let \( \Psi : \pi_n X \rightarrow \pi_n Y \) be an isom.

Define \( f : X^{(n)} \rightarrow Y \) via \( S^n \rightarrow Y \), so that \( \Psi([S^n]) \)
\( f \) "implements" \( \Psi \). Extends over any \( n+1 \) cell \( e^{n+1}_\beta \) by noting that \( f_\#(\partial e^{n+1}_\beta) = \Psi(\partial e^{n+1}_\beta) = 0 \)
and hence \( f|\partial e^{n+1}_\beta \)
extends over \( e^{n+1}_\beta \).

Repeat and then apply Whitehead's Thm.