Lecture 24: Actual computations!

Def: $X$ is $n$-connected if $\pi_i(X, x_0) = 0$ for all $i \leq n$.

$0$-connected = path connected; $1$-conn. = simply connected.

Equivalently, every map from $S^i \to X$ is hom. to a const. map.

Def: $(X, A)$ is $n$-connected if $\pi_i(X, A, x_0) = 0$ for all $x_0 \in A$ and $0 \leq i \leq n$ and also each path component of $A$ contains a pt of $A$.

[Note: About excision.]

Thm: Let $X$ be a CW complex decomposed as a union of two subcomplexes $A$ and $B$ with $C = A \cap B \neq \emptyset$.

If $(A, C)$ is $m$-connected and $(B, C)$ is $n$-connected, then

$$\pi_i(A, C) \xrightarrow{i_*} \pi_i(X, B)$$

is an isomorphism for $i < n + m$ and a surjection for $i = n + m$.

[Query: What does excision say for homology here?]
$SX = X \times [-1, 1] \setminus (X \times \{1\} \cup X \times \{0\})$

$S \cdot S^n = S^{n+1}$

$S$ functor: \(f: X \rightarrow Y\) gives \(Sf: SX \rightarrow SY\)

(from category of top spaces and cont. maps to its)

Leads to:

\[\pi_n (X, x_0) \rightarrow \pi_{n+1} (SX, x_0 \times \{0\})\]

\[\left(S^n, s_o\right) f \rightarrow (X, x_0) \quad \left(S^{n+1}, s_o \times \{0\}\right) \xrightarrow{Sf} \left(SX, x_0 \times \{0\}\right)\]

Freudenthal Suspension Thm: \(X\) is \(n-1\) connected.

Then \(\pi_i: X \xrightarrow{S} \pi_{i+1} SX\) is an isom for \(i < 2n-1\) and onto for \(i = 2n-1\).

Cor: \(\pi_n S^n \cong \mathbb{Z}\) and is gen by \(\text{id}_{S^n}\).

The homotopy class of \(f: S^n \rightarrow S^n\) is determined by its degree.
Pf of Cor: Consider the suspension induced maps:

\[ \mathbb{Z} \cong \pi_1 S^1 \xrightarrow{S} \pi_2 S^2 \cong \pi_3 S^3 \cong \pi_4 S^4 \cong \cdots \]

generated by \( \text{id}_S \) with \( \text{deg} \) by F.S.T. since \( S^n \) is \( (n-1) \) connected.

Since \( \text{deg}: \pi_2 S^2 \to \mathbb{Z} \) is onto, commutativity implies that \( \pi_1 S^1 \to \pi_2 S^2 \) is injective, hence an isomorphism.

---

Pf of F.S.T. from excision: Idea: relate \( \pi_{i+1}(SX) \) to \( \pi_{i+1}(CX, X) \cong \pi_i X \).

Consider:\n
\[ \xymatrix{ C_+ \ar@/^/[rr]^\cong \ar@/_/[rd]_\partial & \pi_{i+1}(C_+X, X) \ar[r]_{i_*} & (SX, C_-) \ar[l] \ar@/^/[ll]_\cong & \pi_{i+1} SX } \]

Claim: this is \( S_* \).

\[ \pi_i X \xrightarrow{C_+} \pi_{i+1}(C_+X, X) \quad \text{via} \quad (f: S^i \to X) \mapsto (CS^i \to CX) \]

Note that \( \partial \circ C_+ = \text{id} \pi_i X \Rightarrow C_+ \) is the inverse isomorphism to \( \partial \).
Claim follows from:

\[ f: \quad S^i \to s_0 \quad \Rightarrow \quad S^i \to (SX, \chi, \chi_0) \quad \text{for } \chi_0 \neq \chi \]

Now apply excision to \( \pi_{i+1}(C_+X, X) \to \pi_{i+1}(SX, C_-X) \)

\( X \text{ is } n-1 \text{ connected } \Rightarrow (C_+X, X) \text{ is } n \text{ connected} \)

\( (C_-X, X) \text{ is } n \text{ connected} \)

\( \Rightarrow i_* \text{ is an isom for } i+1 < 2n \text{ and a surjection for } i+1 = 2n \)
Prop: \((X, A)\) an \(r\)-connected CW pair and \(A\) is \(s\)-connected. Then \(\pi_i(X, A) \to \pi_i(X/A)\) is an isomorphism for \(i \leq r+s\) and onto for \(i = r+s+1\).

\[
\begin{align*}
X \cup CA \cong_{h.e.} X \cup CA/CA \\
\cong_{h.e.} X/A
\end{align*}
\]

Apply excision to \((X, A) \to (X \cup CA, CA)\). Extra dimension comes from \((CA, A)\) being \(s+1\)-connected.