Lecture 16: Second proof of Poincaré Duality

Thm: \( M \) is a closed connected manifold with a PL triangulation \( T \). Then

1. \( H^k(M; \mathbb{F}_2) \cong H_{n-k}(M; \mathbb{F}_2) \)
2. \( H_k(M; \mathbb{F}_2) \times H_{n-k}(M; \mathbb{F}_2) \rightarrow \mathbb{F}_2 \) is nondegenerate.

\[ \alpha \wedge \beta \]

Here, \( \alpha \wedge \beta \) can be defined geometrically or by

\[ \alpha \wedge \beta = (D^{-1}(\alpha) \cup D^{-1}(\beta))(\bar{M}) \]

where \( D: \tilde{H}^*(M) \rightarrow H_*(M) \).

\[ \phi \mapsto [M] \cap \phi \]

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Kinds of manifolds:

- **TOP**: topological manifolds and continuous maps.
- **PL**: Manifolds with PL triangulations and PL maps.
- **DIFF**: Smooth manifolds and smooth maps.

In general, neither are injective or surjective.

\( n \approx \) for dim 1, 2, 3.
Use homology cap product to compute cap product.

\[ \text{Thm: } H^*(\mathbb{R}P^n; \mathbb{F}_2) \cong \mathbb{F}_2(\alpha) / \alpha^{n+1} = 0 \quad | \alpha | = 1 \]

\[ \text{Pf: Let } \alpha_k \text{ be the non-zero elt of } H^*(\mathbb{R}P^n; \mathbb{F}_2). \]

Since \( \mathbb{R}P^{n-1} \hookrightarrow \mathbb{R}P^n \) gives an \( \cong \) on \( H^* \) for \( * < n \),

we need only check that \( \alpha_{n-1} \cup \alpha_1 = \alpha_n \). The

P.D. of \( \alpha_k \) is the non-zero elt of \( H_{n-k}(\mathbb{R}P^n; \mathbb{F}_2) \)

which can be rep. by \( \mathbb{R}P^{n-k} \hookrightarrow \mathbb{R}P^n \). Putting

these in general pos. we see that

\[ \mathbb{R}P^1 \cap \mathbb{R}P^{n-1} = 1 \text{ pt} \]

and so \( \alpha_{n-1} \cup \alpha_1 = \alpha_n \).

\[ \mathbb{R}P^1 \mathbb{R}P^{n-1} \mathbb{R}P^n \]

Inductive proof of Poincaré:

Also P.D. fails for \( \mathbb{R}^n \):

\[ H^0(\mathbb{R}^n; \mathbb{Z}) \neq H_0(\mathbb{R}^n; \mathbb{Z}) \quad \mathbb{Z} \]

Can't subdivide into closed mfdls.
Cohomology with compact supports.

Simplicial version: $X$ is a locally finite complex.\[ C^i_c(X) = \text{cochains taking non-zero values on only finitely many simplices}. \]

Ex: $\mathbb{R}$

$S_0$ in $C^0$ is generated by $\phi: v_i \rightarrow 1$.

$S_0$ in $C^1$ is everything only contains things with sum $= 0$.

Note: Poincaré Duality holds here for $H_c^k$.

de Rahm: $\Omega^k(M)$ smooth $k$-forms\[ \Omega^k_c(M) = \left\{ \alpha \in \Omega^k(M) \mid \exists K^c \in M \text{ with } \alpha = 0 \text{ in } M \setminus K \right\} \]
Singular: \( C^i_c(X) = \{ \varphi \in C^i(X) \mid \exists K_{\varphi}^c \subseteq X \text{ with } \varphi(\sigma) = 0 \ \forall \sigma : \Delta^k \rightarrow X \setminus K \} \)  

[Subcomplex so get] \( \Rightarrow H^i_c(X) \).

\[ \text{Issue:} \]

\[ \text{Want to say that } \varphi \text{ is non-zero only on simplices contained in } K_{\varphi}, \text{ Not a subplex!} \]

Goal: \( H^i_c(X) = \lim_{\rightarrow} H^i(X \setminus K) \)

[Query: How many have seen direct and/or inverse limits?]

\( I = \) partially ordered set where \( \forall \alpha, \beta \in I \ \exists \gamma \) with \( \alpha \leq \gamma, \beta \leq \gamma \).

[Directed graph w/ no directed cycles. Any \( \alpha, \beta \) are "below" some other vertex.]
Groups: $G_\alpha$ for each $\alpha \in I$

$f_{\alpha \beta} : G_\alpha \to G_\beta$ for each $\alpha \leq \beta$

If $\alpha \leq \beta \leq \gamma$ have $f_{\alpha \gamma} = f_{\beta \gamma} \circ f_{\alpha \beta}$

[Called a directed system of groups.]

Direct Limit:

$$\lim_{\to} G_\alpha = \prod_{\alpha \in I} G_\alpha \bigg/ \{ a \in G_\alpha \sim b \in G_\beta \mid \text{if } \exists \gamma \text{ with } \alpha \leq \gamma, \beta \leq \gamma \text{ and } f_{\alpha \gamma}(a) = f_{\beta \gamma}(b) \text{ in } G_\gamma \}$$

[Check that this an equiv. relation. Can add.]

Set $[a] + [b] = [f_{\alpha \gamma}(a) + f_{\beta \gamma}(b)]$ where $\alpha, \beta \leq \gamma$.

[Alternatively] $= \bigoplus_{\alpha} G_\alpha \bigg/ \text{subgrp gen by } a - f_{\alpha \beta}(a) \forall a \in G_\alpha \forall \beta > \alpha.$

$E_\infty$: $I = \mathbb{Z}$

$G_\alpha = \mathbb{Z}^\alpha$

$$\lim_{\to} G_\alpha = \bigoplus_{\alpha = 1}^{\infty} \mathbb{Z}$$

$f_{\alpha \beta} : \mathbb{Z}^\alpha \to \mathbb{Z}^\beta$

$$(x_1, \ldots, x_\alpha) \mapsto (x_1, \ldots, x_\alpha, 0, \ldots, 0)$$
$X$ top space

$I = \{ \text{cpt subsets of } X \} \leq \text{containment of subsets}$

$G_K = H^n(X \setminus K)$

If $K \subseteq K'$ then have

$H^n(X \setminus K) \rightarrow H^n(X \setminus K')$

[Typically not injective or surjective.]

**Prop:** $H^n_c(X) \cong \lim_{K \in \text{opt}} H^n(X \setminus K)$

**Pf:** First note that $H^n(X \setminus K) \rightarrow H^n_c(X)$ since $\phi \in C(X \setminus K)$ vanishes on any $\sigma$ in $X \setminus K$.

[Compatible with inclusions, so get ] $\lim H^n(X \setminus K) \rightarrow H^n_c(X)$

**Surjective:** $[\phi] \in H^n_c(X)$ is in the image of $H^n(X \setminus K)_{\phi}$

**Injective:** If $[\phi] \in H^n(X \setminus K)$ is 0 in $H^n_c(X)$, then $\phi = \delta \psi$ where $\psi$ is supported on $K \psi \supset K_{\phi}$

In particular, $[\phi] = 0$ in $H^n(X \setminus K \psi)$.